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The Statistics of Honeycomb and Triangular Lattice. II.

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4) Representations of Algebra.[1]

In chapter 2, we have studied the multiplication table of the algebra $\boldsymbol{H}_r$ of order 70. Here we want to get its representations. It is easily verified that the 'center' [the subalgebra which commutes with all its elements] is formed of the following five elements.

$$I=(1/2)(1-(-)^r C),\quad S=j+k+j_s+k_s,\quad T=jk+j_sk_s+j_sj+k_sk+j_sk+jk_s,$$

$$W=jk_sj_s+jkj_s+jkk_s+j_skk_s,\quad P=jkj_sk_s.$$

The multiplication table is

	I	S	T	W	P
S	S	$2(T-2I)$	$3(W-S)$	$2(2P-T)$	$-W$
T	T	$3(W-S)$	$6(I-P)-4T$,	$3(S-W)$	T
W	W	$2(2P-T)$	$3(S-W)$	$2(T-2I)$	$-S$
P	P	$-W$	T	$-S$	I

And from this, the orthogonal idempotents which are the unities of the two sided ideals are found as follows

$$A=(I-P)/4+i(S+W)/8$$

$$A^*=(I-P)/4-i(S+W)/8$$

$$B=3(I+P)/8+T/8$$

$$\bar{C}=(I+P)/16-T/16-i(W-S)/16$$

$$C^*=(I+P)/16-T/16+i(W-S)/16$$

where $\qquad I=A+A^*+B+\bar{C}+C^*.$

Let us first consider the ideal having the unity A. This simple ideal is $A\cdot\boldsymbol{H}_r$ which is formed by multiplying A by every element of the algebra $\boldsymbol{H}_r$. As easily seen, $A\cdot\boldsymbol{H}_r$ is a simple algebra of order 16. By Wedderburn's theorem it is isomorphic to the total matric algebra of degree 4.

The unity A is resolved into the sum of 4 primitive idempotents

$$A=EE'A+E''EA+E_sE_s'A+E_sE'''A.$$

And as $EE'E''=E_sE_s'E'''=0$, and $E_sAEA=0$, these four idempotents are pairwise orthogonal. Then this ideal may be resolved into four simple left ideals each having four linearly independent elements. For example, left ideal made by $EE'A$ is formed by

$$\textcircled{1}=EE'A=1/16(I-P+jk+jk_s+jj_s-j_sk_s-kk_s-j_sk)$$
$$+i/16(j_s+k+k_s-j+jkj_s+jkk_s+jj_sk_s-j_skk_s),$$
$$\textcircled{2}=X\textcircled{1}=1/8(X-Yk_s-Yj_s-Xj_sk_s)+i/8(Y+Xk_s+Xj_s-Yj_sk_s),$$
$$\textcircled{3}=U\textcircled{1}=1/8(U-Vk-Vj_s-Uj_sk)+i/8(V+Uj_s+Uk-Vj_sk),$$
$$\textcircled{4}=M'\textcircled{1}=1/8(M'-N'k-N'k_s-M'kk_s)+i/8(N'+M'k+M'k_s-N'kk_s),$$
$$X_s\textcircled{1}=0,\quad U_s\textcircled{1}=0,\quad M\textcircled{1}=0.$$

As the irreducible representations by this left ideal, we can get

$$XA=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad YA=\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad UA=\begin{pmatrix}0&0&-1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$$

Or to save the space, we introduce the matrix e_{ij} all elements of which are zero except the (i, j) element which is 1. It is clear that these e_{ij} have the following properties

$$e_{ij}e_{jk}=e_{ik},\quad e_{ij}e_{kl}=0\,(j\neq k).$$

These are the bases of the total matric algebra of degree 4. Then

$$\begin{array}{llll}XA=-e_{12}+e_{21}, & YA=-i(e_{12}+e_{21}), & EA=e_{11}+e_{22}, & UA=-e_{13}+e_{31}\\ VA=-i(e_{13}+e_{31}), & E'A=e_{11}+e_{33}, & M'A=-e_{14}+e_{41}, & N'A=-i(e_{14}+e_{41})\\ E'''A=e_{11}+e_{44}, & X_sA=-e_{34}+e_{43}, & Y_sA=i(e_{34}+e_{43}), & E_sA=e_{33}+e_{44}\\ U_sA=-e_{24}+e_{42}, & V_sA=i(e_{24}+e_{42}), & E_s'=e_{22}+e_{44}, & MA=e_{23}-e_{32}\\ NA=i(e_{23}+e_{32}), & E''A=e_{22}+e_{33}, & A=e_{11}+e_{22}+e_{33}+e_{44}. & \end{array}$$

The representations of other elements are omitted. The representations by the other left ideals are similar to those obtained. The simple ideal $\boldsymbol{H}_rA^*$ is also of order 16 and so isomorphic to the total matric algebra of degree 4. Hence $\boldsymbol{H}_rA$ is isomorphic to $\boldsymbol{H}_rA^*$. It must be noticed that in this ideal (XA, YA) are orthogonal to (X_sA, Y_sA), that is $XX_sA=Y_sYA=XY_sA=YX_sA=0$, And iXA, iX_sA can have the simultaneous eigenvalues $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$. This fact may be useful in a later chapter.

As the next step, we shall study the ideal $\boldsymbol{H}_rB$.

If we put, remembering of 6) of chapter 2

$$2x_1 = XX_s - YY_s = -(UU_s - VV_s)$$
$$2x_2 = XX_s + YY_s = -(MM' + NN')$$
$$2x_3 = -(MM' - NN') = UU_s + VV_s,$$

then it is easily seen that x_1, x_2, x_3 are orthogonal and commutative each other.

$$x_i x_k = x_k x_i = 0 \quad (i \neq k,\ 1, 2, 3).$$

Further we put

$$\varepsilon_1 = x_1^2 = 1/8(I + P + T - 2jj_s - 2kk_s) = E'EB = E_s'E_sB$$
$$\varepsilon_2 = x_2^2 = 1/8(I + P + T - 2jk_s - 2kk_s) = EE'B = E_sE''B$$
$$\varepsilon_3 = x_3^2 = 1/8(I + P + T - 2jk - 2j_sk_s) = E'E'''B = E_s'E''B.$$

It is easily proved that

$$\varepsilon_i^2 = \varepsilon_i \qquad (i = 1, 2, 3).$$
$$\varepsilon_i\varepsilon_k = \varepsilon_k\varepsilon_i = 0. \qquad (i \neq k,\ 1, 2, 3).$$

Then we can dissolve B as the sum of 6 primitive idempotents

$$B = 1/2(\varepsilon_1 - x_1) + 1/2(\varepsilon_1 + x_1) + 1/2(\varepsilon_2 - x_2) + 1/2(\varepsilon_2 + x_2)$$
$$+ 1/2(\varepsilon_3 - x_3) + 1/2(\varepsilon_3 + x_3).$$

These 6 idempotents are pairwise orthogonal. As we can see easily the ideal $\boldsymbol{H}_r B$ has 36 linearly independent elements. So that this is isomorphic to the total matric algebra of degree 6. The left ideal, for example $\boldsymbol{H}_r B(\varepsilon_1 - x_1)$ becomes the basis of the irreducible representation of the algebra $\boldsymbol{H}_r B$.

$$① = \varepsilon_1 - x_1 = e_{11}$$
$$② = X(①) = X_s(①) = e_{21}$$
$$= 1/4(X + Xj_sk_s + Yj_s - Yk_s) + 1/4(X_s + X_sjk + Y_sj - Y_sk)$$
$$③ = U(①) = -V_s(①) = e_{31}$$
$$= 1/4(U + Uj_sk - Vk + Vj_s) + 1/4(U_s + U_sjk_s + V_sj - V_sk_s)$$
$$④ = XY(①) = X_sY_s(①) = UV(①) = U_sV_s(①) = e_{41}$$
$$⑤ = Y(①) = -Y_s(①) = e_{51}.$$
$$= 1/4(Y + Yj_sk_s - Xj_s + Xk_s) + 1/4(-Y_s - Y_sjk + X_sj - X_sk)$$
$$⑥ = V(①) = V_s(①) = e_{61}$$
$$= 1/4(V + Vj_sk + Uk - Uj_s) + 1/4(V_s + V_sjk_s - U_sj + U_sk_s).$$

These 6 elements are the bases of the left ideal $\boldsymbol{H}_r B(\varepsilon_1 - x_1)$. The representations

obtained by using the basis of the total matric algebra of degree 6, are as follows.

$$XB=-e_{12}+e_{21}+e_{45}-e_{54} \qquad X_sB=-e_{12}+e_{21}-e_{45}+e_{54}$$

$$YB=-e_{15}+e_{51}+e_{24}-e_{42} \qquad Y_sB=-e_{51}+e_{15}+e_{24}-e_{42}$$

$$XYB=-e_{14}+e_{41}+e_{52}-e_{25} \qquad X_sY_sB=-e_{14}+e_{41}+e_{25}-e_{52}$$

$$EB = e_{11}+e_{22}+e_{33}+e_{44}=E_sB$$

$$UB=-e_{13}+e_{31}+e_{46}-e_{64} \qquad U_sB=e_{13}-e_{31}+e_{46}-e_{64}$$

$$VB=-e_{16}+e_{61}+e_{34}-e_{43} \qquad V_sB=-e_{16}+e_{61}-e_{34}+e_{43}$$

$$UVB=-e_{14}+e_{41}-e_{36}+e_{63} \qquad U_sV_sB=-e_{14}+e_{41}+e_{36}-e_{63}$$

$$E'B = e_{11}+e_{33}+e_{44}+e_{66}=E_s'B$$

$$MB = e_{23}-e_{32}+e_{56}-e_{65} \qquad M'B=e_{32}-e_{23}+e_{56}-e_{65}$$

$$NB = e_{26}-e_{62}+e_{35}-e_{53} \qquad N'B=e_{62}-e_{26}+e_{35}-e_{53}$$

$$MNB = e_{25}-e_{52}+e_{63}-e_{36} \qquad M'N'B=-e_{25}+e_{52}-e_{36}+e_{63}$$

$$E''B = e_{22}+e_{33}+e_{55}+e_{66}=E'''E$$

$$B=e_{11}+e_{22}+e_{33}+e_{44}+e_{55}+e_{66}.$$

The representations by other five left ideals are similar to these. The ideals $\boldsymbol{H}_r\bar{C}$ and $\boldsymbol{H}_rC^*$ are of order 1, so that isomorphic to the complex number. The elements except the commutable subalgebra (I, j, k, j_s, k_s) have representations 0. The algebra $\boldsymbol{H}_r$ is thus divided into the direct sum of five simple ideals.

$$\boldsymbol{H}_r=\boldsymbol{H}_rA+\boldsymbol{H}_rA^*+\boldsymbol{H}_rB+\boldsymbol{H}_r\bar{C}+\boldsymbol{H}_rC^*.$$

The sum of these orders $16+16+36+1+1=70$ coincides to the order of $\boldsymbol{H}_r$.

5) Solution of the Eigenvalue Problem.

In this chapter we want to diagonalise the operator $\bar{V}_r$ and then V. As $\bar{V}_r$ belongs to the algebra $\boldsymbol{H}_r$, we can resolve it to the direct sum of five parts belonging to each ideal.

$$\bar{V}_r=\bar{V}_rA+\bar{V}_rA^*+\bar{V}_rB+\bar{V}_r\bar{C}+\bar{V}_rC^*.$$

And these five parts are pairwise orthogonal, and commutative. So we can diagonalise $\bar{V}_r$ separately on each ideal.

a) On the ideal $\boldsymbol{H}_rA$.

As X_sA, X_rA, E_rA, E_sA are commutative and linearly independent and $X_rAX_sA=0$, we can transform $\bar{V}_rA$ into the form

$$\exp(i\gamma_rX_r)\exp(i\gamma_sX_s)A.$$

Then we want to calculate γ_r, γ_s.

As the characteristic equation is invariant by any similarity transformation and iX_rA, iX_sA can take the eigenvalues (0, 1), (0, −1) (1, 0) and (−1, 0) simultaneously, we have

$$\text{Trace } (\bar{V}_rA) = e^{\gamma_r} + e^{-\gamma_r} + e^{\gamma_s} + e^{-\gamma_s}$$

$$= 2(\text{ch } \gamma_r + \text{ch } \gamma_s).$$

Sum of products of every two eigenvalues is

$$= 2(1 + 2\text{ch } \gamma_r \text{ ch } \gamma_s).$$

These are the invariants of similarity transformations.

To calculate the left hand side, we must make the matrix representations of degree 4 of $\bar{V}_rA$. Developing as

$$\exp\{-iH(X_r^* + X_s^*)\} \cdot A = A \text{ ch } H - i(X_r^* + X_s^*)A \cdot \text{sh } H, \text{ etc.}$$

(Here we must notice that $(X_r^* + X_s^*)$ and $(U_r^* + U_s^*)$ are commutative with each other.)

By making the matrix product of these, we can get the matrix representation of $\bar{V}_rA$. From it we can easily calculate the left hand side of (5.3) and (5.4). And they become

$$4(\text{ch } 2H \text{ ch } 2H^+ - \sin^2\varphi_r) = 2(\text{ch } \gamma_r + \text{ch } \gamma_s)$$

$$2 + 4(\text{ch}^2 2H + \text{ch}^2 2H^+ - 2\cos^2\varphi_r + 1) = 2(1 + 2 \text{ ch } \gamma_r \text{ ch } \gamma_s)$$

where $(\text{ch } 2H - 1)(\text{ch } 2H^+ - 1) = 1$ or $\text{ch } 2H^+ = \text{ch}^2 2H^* + \text{sh}^2 2H^* \cdot \text{ch } 2H.$

Then we can get $\text{ch}\gamma_r$ and $\text{ch}\gamma_s$ as the two roots of the quadratic equation

$$y^2 - 2y(\text{ch } 2H \text{ ch } 2H^+ - \sin^2\varphi_r) + \text{ch}^2 2H + (\text{ch}^2 2H^+ - 2\cos^2\varphi_r + 1) = 0.$$

That is

$$\text{ch } \gamma_r = \text{ch } 2H \text{ ch } 2H^+ - \sin^2\varphi_r - \cos\varphi_r(\text{sh}^2 2H \text{ sh}^2 2H^+ - \sin^2\varphi_r)^{1/2} \tag{5.5}$$

$$\text{ch } \gamma_s = \text{ch } 2H \text{ ch } 2H^+ - \sin^2\varphi_s - \cos\varphi_s(\text{sh}^2 2H \text{ sh}^2 2H^+ - \sin^2\varphi_s)^{1/2}$$

$$(\varphi_r = r\pi/n, \quad \varphi_s = \pi s/n = \pi + r\pi/n).$$

Using these γ_r, γ_s, we can write the transformed eigenoperator as

$$\exp(i\gamma_rX_r + i\gamma_sX_s)A.$$

Similarly for the ideal $\boldsymbol{H}_rA^*$

$$\exp(i\gamma_rX_r + i\gamma_sX_s)A^*.$$

b) On the ideal $\boldsymbol{H}_rB$.

From the representation (4.6), it is easily seen that the each factor matrix of $\bar{V}_rB$ may be considered as the rotation of imaginary angle in six dimensional

soace. The product $\bar{V}_r B$ is the resultant rotation which fixes the sixth axis. Then this is able to be brought to the canonical form by suitable transformation

$$\cos\theta(e_{11}+e_{22})+\sin\theta(e_{12}-e_{21})+e_{33}+e_{66}+\cos\varphi(e_{44}+e_{55})+\sin\varphi(e_{45}-e_{54})$$
$$=\exp[\theta/2(X_r+X_s)+\varphi/2(+X_r-X_s)]B=\exp(i)(\bar{\gamma}_r X_r+\bar{\gamma}_s X_s)B.$$
$$(i\bar{\gamma}_r=(\theta+\varphi)/2,\ i\bar{\gamma}_s=(\theta-\varphi)/2).$$

We can see from the representation (4.6) that in this ideal, iX_rB iX_sB can take the eigenvalue sets $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$, $(0,0)$ and $(0,0)$ simultaneously. Then the eigenvalues of $\exp(i\bar{\gamma}_r X_r+i\bar{\gamma}_s X_s)B$ are

$$e^{\bar{\gamma}_r+\bar{\gamma}_s},\quad e^{\bar{\gamma}_r-\bar{\gamma}_s},\quad e^{-\bar{\gamma}_r+\bar{\gamma}_s},\quad e^{-\bar{\gamma}_r-\bar{\gamma}_s},\qquad 1,1.$$

The sum of these $=2[\mathrm{ch}(\bar{\gamma}_r+\bar{\gamma}_s)+\mathrm{ch}(\bar{\gamma}_r-\bar{\gamma}_s)+1]$.
The sum of products of two of these $=4(\mathrm{ch}\,\bar{\gamma}_r+\mathrm{ch}\,\bar{\gamma}_s)^2-1$.
These become by actual calculation of matrix representation of $\bar{V}_r B$

$$\begin{aligned}&2(2\mathrm{ch}\,\bar{\gamma}_r\mathrm{ch}\,\bar{\gamma}_s+1)=2+4(\mathrm{ch}^2 2H+\mathrm{ch}^2 2H^+ +1-2\cos^2\varphi_r)\\&4(\mathrm{ch}\,\bar{\gamma}_r+\mathrm{ch}\,\bar{\gamma}_s)^2-1=16(\mathrm{ch}\,2H\,\mathrm{ch}\,2H^+)^2-1-32\mathrm{ch}\,2H\,\mathrm{ch}\,2H^+\sin^2\varphi_r-16\sin^4\varphi_r.\end{aligned}\tag{5.6}$$

From (5.5), (5.6) it is clear that $\bar{\gamma}_r=\gamma_r$, $\bar{\gamma}_s=\gamma_r$.

c) On the ideal $\boldsymbol{H}_r C$ and $\boldsymbol{H}_r C^*$, it is clear that

$$\bar{V}_r(\bar{C}+C^*)=\bar{C}+C^*=\exp(i\gamma_r X_r+i\gamma_s X_s)(\bar{C}+C^*).$$

So that by adding these

$$\exp(i\gamma_r X_r+i\gamma_s X_s)(A+A^*+B+\bar{C}+C^*)=\exp(i\gamma_r X_r+i\gamma_s X_s).$$

This shows that $\bar{V}_r$ can be transformed to the form which diagonalises X_r, X_s simultaneously by some operator S_r

$$S_r\bar{V}_r S_r^{-1}=\exp(i\gamma_r X_r+i\gamma_s X_s).$$

As for $\bar{V}_0$, it is easily seen that

$$S_0\bar{V}_0 S_0^{-1}=\exp[-i(H-H^+)X_0+i(H+H^+)X_n].$$

And for $V_{n/2}$

$$S_{n/2}\bar{V}_{n/2}S_{n/2}^{-1}=\exp[i\gamma_{n/2}X_{n/2}].$$

Putting $S=S_0S_1S_2\cdots\cdots S_{n/2}$

$$\begin{aligned}SVS^{-1}=(2\mathrm{sh}2H)^n\exp[&-i(H-H^+)X_0+i\gamma_1X_1+i\gamma_2X_2+\cdots\cdots\\&+i\gamma_{n-1}X_{n-1}+i(H+H^+)X_n].\end{aligned}$$

Dividing to the subspaces where C has the eigenvalue 1 and -1.

$$\frac{1+C}{2}SVS^{-1}=\frac{1+C}{2}(2\mathrm{sh}2H)^n\exp(i\gamma_1X_1+i\gamma_3X_3+\cdots\cdots+i\gamma_{n-1}X_{n-1}).$$

$$\frac{1-C}{2}SVS^{-1}=\frac{1-C}{2}(2\operatorname{sh}2H)^n\exp(-i(H-H^+)X_0+i\gamma_2X_2+\cdots+i(H+H^+)X_n).$$

Or in (ξ, η) representation they become

$$\frac{1+C}{2}SVS^{-1}=(2\operatorname{sh}2H)^n\frac{1+C}{2}\exp\left\{i\gamma_1\frac{\xi_1\eta_1+\xi_1^+\eta_1^+}{2}+i\gamma_3\frac{\xi_3\eta_3+\xi_3^+\eta_3^+}{2}+\cdots\cdots\right.$$
$$\left.+i\gamma_{n-1}\frac{\xi_{n-1}\eta_{n-1}+\xi_{n-1}^+\eta_{n-1}^+}{2}\right\}$$

$$\frac{1-C}{2}SVS^{-1}=(2\operatorname{sh}2H)^n\frac{1-C}{2}\cdot\exp\left\{-i(H-H^+)\xi_0\eta_0+i\gamma_2\frac{\xi_2\eta_2+\xi_2^+\eta_2^+}{2}+\cdots\cdots\right.$$
$$\left.+i(H+H^+)\xi_n\eta_n\right\}.$$

i) In the subspace where C has the eigenvalue 1.

As $(\xi_r\eta_r)^2=-1$, the eigenvalue of $\xi_r\eta_r$ is i or $-i$. And

$$(\xi_1\eta_1),\ (\xi_1^+\eta_1^+),\ (\xi_3\eta_3),\ (\xi_3^+\eta_3^+)\cdots\cdots(\xi_{n-1}\eta_{n-1}),\ (\xi_{n-1}^+\eta_{n-1}^+)$$

are commutative and independent each other. So they can take the eigenvalues $\pm i$ independently only obeying the restriction that

$$C=i^n(\xi_1\eta_1)(\xi_1^+\eta_1^+)(\xi_3\eta_3)(\xi_3^+\eta_3^+)\cdots\cdots(\xi_{n-1}\eta_{n-1})(\xi_{n-1}^+\eta_{n-1}^+)=1.$$

The maximum eigenvalue corresponds to the case where

$$(\xi_1\eta_1)=(\xi_1^+\eta_1^+)=(\xi_3\eta_3)=(\xi_3^+\eta_3^+)=\cdots\cdots=(\xi_{n-1}^+\eta_{n-1}^+)=-i$$

that is

$$\lambda_m=(2\operatorname{sh}2H)^n\exp(\gamma_1+\gamma_3+\cdots\cdots+\gamma_{n-1}).$$

ii) In the subspace where C has the eigenvalue -1

$$(\xi_0\eta_0),\ (\xi_2\eta_2),\ (\xi_2^+\eta_2^+),\ (\xi_4\eta_4),\cdots\cdots(\xi_{n-2}^+\eta_{n-2}^+),\ (\xi_n\eta_n)$$

are also commutative and independent each other. So they can take the eigenvalue $\pm i$ independently only obeying the restriction that

$$C=i^n(\xi_0\eta_0)(\xi_2\eta_2)(\xi_2^+\eta_2^+)\cdots\cdots(\xi_{n-2}^+\eta_{n-2}^+)(\xi_n\eta_n)=-1.$$

The maximum in this subspace, that is next to the maximum in the whole space, corresponds to the case where

$$-(\xi_0\eta_0)=(\xi_2\eta_2)=(\xi_2^+\eta_2^+)=\cdots\cdots=(\xi_{n-2}^+\eta_{n-2}^+)=(\xi_n\eta_n)=-i$$

that is

$$\lambda_{(-)}=(2\operatorname{sh}2H)^n\exp((H-H^+)+\gamma_2+\gamma_4+\gamma_{n-2}+(H+H^+)).$$

6) The Thermodynamical Properties of Large Crystal.

In the large crystal, $n\to\infty$, the sum of (5.7) (5.8) will be replaced by the integral. That is

$$\log \lambda_m - n\log(2\mathrm{sh}\,2H) = \frac{n}{2\pi}\int_0^\pi \gamma d\varphi \tag{6.1}$$

$$\log \lambda_{(-)} - n\log(2\mathrm{sh}\,2H) = \frac{n}{2\pi}\int_0^\pi \gamma d\varphi \qquad H>H^+ \tag{6.2}$$

$$= 2(H-H^+) + \frac{n}{2\pi}\int_0^\pi \gamma d\varphi \qquad H<H^+$$

From (6.1) $\qquad f_h(H) \doteqdot \lambda_m^N$.

And (6.1) (6.2) show that

$$\lim_{n\to\infty}\frac{\lambda_m}{\lambda_{(-)}} = \begin{cases} 1 & H>H^+ \\ \exp 2(H-H^+) & H<H^+ \end{cases}$$

The largest eigenvalue and the next to it coincide below the temperature defined by $H=H^+$, and have some difference above it. This shows that the temperature defined by $H=H^+$, that is $\mathrm{ch}\,2H=2$, is the Curie point of the honeycomb lattice. The coincidence of the eigenvalues below the critical temperature shows that there is long range order in the crystal. To evaluate the integral (6.1) we put

$$z = \mathrm{ch}\,2H\,\mathrm{ch}\,2H^+ - \sin^2\varphi - \cos\varphi\,(\mathrm{sh}^2 2H\,\mathrm{sh}^2 2H^+ - \sin^2\varphi)^{1/2}$$

$$I = \int_0^\pi \mathrm{ch}^{-1} z d\varphi = \int_0^\pi \log(z+\sqrt{z^2-1})d\varphi.$$

As shown from this expression, $z=1$ is the branching point of the integrand. This point is reached when $H=H^+$ and $\varphi=0$. When n is finite, $\varphi_r=\pi r/n$ can not reach this value. So as expected, the singularity occurs when $n\to\infty$. The critical temperature located by $\mathrm{ch}\,2H=2$ is identical with the value given by the 'dual and star-triangle' transformations.[2]

To see the character of singularity, we shall integrate the integral I assuming $|H-H^+|$ is small enough, near the interval $\varphi=0$. As z outside the radical sign gives no effect on the character of singularity, we put it 1. And as $\sqrt{z^2-1}$ is small, developing the logarithm in power series, we confine our attention on the first term

$$I \cong \int_0 \sqrt{z^2-1}\,d\varphi$$

$$\cong \int_0 \sqrt{(2H-2H^+)^2+4\varphi^2/3}\,d\varphi$$

$$= \sqrt{3}\,(H-H^+)^2 \log\frac{1}{|H-H^+|} + \text{analytic function}$$

Let λ_1 be the partition function per atom, then

$$\lambda_1=\lambda_m^{1/2n}.$$

Thus the sigularity of $\log \lambda_1$ near $H=H^+$ is given by

$$\log \lambda_1=\frac{\sqrt{3}}{4\pi}(H-H^+)^2\log\frac{1}{|H-H^+|}+\text{analytic function.}$$

This does not give the discontinuity in the partition function and internal energy. But the specific heat, given by

$$C=NkH_c^2\frac{d^2\log\lambda_1}{dH^2}$$

becomes logarithmically infinite at the Curie point.

$$C/Nk=\frac{2\sqrt{3}}{\pi}H_c^2\log\frac{1}{|H-H^+|}+\text{analytic functions.}$$

In the former approximation theory of order-disorder transition, the specific heat shows a finite jump at the Curie point asymmetrically on both sides of it. But this theory shows that, contrary to it, the specific heat becomes logarithmically infinite at the Curie point as in the Onsager's case.

7) Case of Antiferromagnetism.

In the case of antiferromagnetism, where we assume J is negative, that is H is negative, we must put $(-H)$ instead of H, where H is positive. By definition

$$\operatorname{ch}2(-H)^+=\operatorname{ch}2H^+,\quad \operatorname{sh}2(-H)^+=\ -\operatorname{sh}2H^+.$$

(If we take the negative sign.)

If we put these in (5.7) the partition function remains unchanged as the function of H. Then as indicated in the former chapter the singularity occurs at $\operatorname{ch}2H=2$, and the specific heat becomes logarithmically infinite at this point.

8) Case of Triangular Lattice.

The triangular net is the so-called dual net of the honeycomb lattice. Then the partition function of the former is associated with the latter by the relation

$$f_t(H)=2^{t-1-s/2}(\operatorname{sh}2H)^{s/2}f^*(H^*) \qquad (8.1)$$

where t is the number of atoms in the triangular net, s is the number of interacting lines. We have known the functional form of f^* in the preceding chapter. So we may put H^* instead of H in f_h of (6.2).

Then

$$\operatorname{ch} 2H^* \operatorname{ch} 2(H^*)^+ = e^{4H}\left(\frac{1}{2}+\frac{2}{e^{4H}-1}\right)=\frac{1}{2}(e^{4H}e^{4H_0}-1).$$

Here defined H_0 as

$$(e^{4H}-1)(e^{4H_0}-1)=4.$$

And

$$\operatorname{sh} 2H^* \operatorname{sh} 2(H^*)^+ = e^{2H}e^{2H_0}.$$

If we put

$$z(H^*)=(1/2)(e^{4H}e^{4H_0}-1)-\sin^2\varphi-\cos\varphi(e^{4H}e^{4H_0}-\sin^2\varphi)^{1/2}$$

then

$$f^*(H^*)=\left[(2\operatorname{sh} 2H^*)^n \exp\frac{n}{2\pi}\int_0^\pi \operatorname{ch}^{-1} z(H^*)d\varphi\right]^{\nu}.$$

The singularity occurs at $z=1$, which is only achieved when $\varphi=0$ and $H=H_0$, that is $e^{4H}=e^{4H_0}=3$.

Then to see the character of singularity we shall integrate it in the neighbor of $H=H_0$ and $\varphi=0$.

$$z(H^*)=(1/2)(e^{4H}+e^{4H_0}+2)-\sin^2\varphi-\left(1-2\sin^2\frac{\varphi}{2}\right)e^{2H}e^{2H_0}(1-\sin^2\varphi/18)$$

$$=1+(1/2)(e^{2H}-e^{2H_0})^2+(2/3)\varphi^2$$

$$\int_0 \operatorname{ch}^{-1} z(H^*)d\varphi=\int_0 \log(z+\sqrt{z^2-1})d\varphi\cong\int_0 \sqrt{z^2-1}\,d\varphi$$

$$\cong\int_0 \sqrt{(e^{2H}-e^{2H_0})^2+(4/3)\varphi^2}\,d\varphi$$

$$=-\frac{\sqrt{3}}{4}(e^{2H}-e^{2H_0})^2\log|e^{2H}-e^{2H_0}| + \text{analytic function.}$$

At $H=H_0$, the specific heat becomes logarithmically infinite as in the case of square net and honeycomb lattice.

9) Antiferromagnetic Case of Triangular Lattice.

To get the partition function for the antiferromagnetic case, we must put $(-H)$ for H in (8.1), (8.2)

$$f_t(-H)=2^{t-1-s/2}(\operatorname{sh} 2H)^{s/2}(2\operatorname{sh} 2H^*)^{nN}\cdot\exp\frac{nN}{2\pi}\int_0^\pi \operatorname{ch}^{-1} z(-H)d\varphi$$

where

$$z(-H)=(1/2)(-e^{-4H}(2+e^{4H_0})-1)-\sin^2\varphi-i\cos\varphi\sqrt{e^{-4H}(2+e^{4H_0})+\sin^2\varphi},$$

$$e^{4(-H)_0}=-(2+e^{4H_0}).$$

As $z(-H)\neq\pm1$ the singularity of $\mathrm{ch}^{-1}z$ occurs when $\varphi=0$ and $H\to\infty$. To see the character of this singularity we shall put

$$e^{-4H}=x^2,\ e^{4H_0}=1+4x^2,\quad x,\ \varphi;\quad \text{small}$$

Then
$$z(-H)=-\frac{1}{2}-\frac{3}{2}x^2-\sin^2\varphi\ -i\sqrt{3x^2+\sin^2\varphi}$$

$$|z+\sqrt{z^2-1}\,|=1-(4/\sqrt{3}\,)\sqrt{3x^2+\varphi^2}$$

$$\log\,(z+\sqrt{z^2-1})=\log\{1-(4/\sqrt{3}\,)\sqrt{3x^2+\varphi^2}\}+i\cdot\mathrm{Arg}(z+\sqrt{z^2-1})$$

$$I=\int_0\log\,(z+\sqrt{z^2-1})d\varphi=4/\sqrt{3}\int_0\sqrt{3x^2+\varphi^2}\,d\varphi$$

$$=2\sqrt{3}\,x^2\log\frac{1}{|x|}+\text{analy. func.}$$

$$=2\sqrt{3}\,e^{-4H}(2H)\ +\text{analy. func. of } x.$$

At absolute zero $H\to\infty$, the specific heat clearly tends to zero. There are no phase change in this case.

In conclusion, the author wishes to express his heartful thanks to Prof. K. Husimi for his kind instructions and encouragement throughout this work. And also the author thanks to the members of the Husimi-Laboratory for their kind discussion with him.

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On the Nature of τ-Mesons. II.

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III. The Decay of $\tau \rightarrow \pi + \gamma$.

The another possibility of the decay of τ-meson is $\tau \rightarrow \pi + \gamma$. The procedure can be translated from II by such slight modification that a photon, denoted by γ, stands instead of a π^0-meson. Accordingly, the coupling with a nucleon is modified as $\gamma_i \tau_p$ instead of $\gamma_M \tau_3$ or γ_M, where $\tau_p = \frac{1}{2}(1+\tau_3)$. For the sake of τ_p, containing 1 and τ_3, there is no selection rule as in II § 2, 1. But if both τ- and π-mesons are neutral, we obtain the selection rule: forbidden if $N_\pi + N_\tau =$ odd.[25] Further we must remark that the gauge problem takes place. Remarking such points we put forth the calculation of the decay probability.

§ 1. Fundamental formulation.

The interaction Hamiltonian describing the system composed of π and τ-mesons, photons and nucleons consists of the following parts:

$$H = H(p, \gamma) + H(n, \pi) + H(\pi, \gamma) + H(p, \pi, \gamma) + [\pi\tau] \tag{3.1}$$

p, n, π, τ and γ in arguments of H represent the proton, nucleon, π-meson, τ-meson and photon. $[\pi\tau]$ means the interchange of π and τ in the preceding three parts. Denoting the wave functions of nucleon, photon, π meson and τ-meson as φ, A_i, U and $U_{ij} = \partial_i U_j - \partial_j U_i$, and V and $V_{ij} = \partial_i V_j - \partial_j V_i$, respectively, the Hamiltonians for respective interactions are represented as follows:

$$H(p, \gamma) = -ie\varphi^+ \gamma_i \tau_p \varphi A_i, \tag{3.2}$$

$$H(\pi, n) = igU_L \varphi^+ \gamma_L \tau_{pN} \varphi + conj. \tag{3.3}$$

where $g(f)$ is the coupling constant between meson and nucleon, and

$$H(\pi, \gamma) = ieA_i J_i \tag{3.4}$$

where J_i means the current produced by a meson,

34) c. f. S. Kanesawa and S. Tomonaga, Prog. Theor. Phys. 2 (1947), 101.

$$J_i = U^*(\overrightarrow{\partial}_i + \overleftarrow{\partial}_i)U \tag{3.5}$$

for scalar or pseudoscalar meson and

$$J_i = U_j^* U_{ij} - U_j U_{ij}^* \tag{3.6}$$

for vector or pseudovector meson. The Hamiltonian containing three particles simultaneously is necessary only for the following cases.

$$H(p,\pi,\gamma) = -\left(\frac{cg}{\mu_\pi}\right)U^*(A_i\varphi^+\tau_{Np}(\gamma_5)\gamma_i\varphi - A_4\varphi^+\tau_{Np}(\gamma_5)\gamma_4\varphi) - conj. \tag{3.7}$$

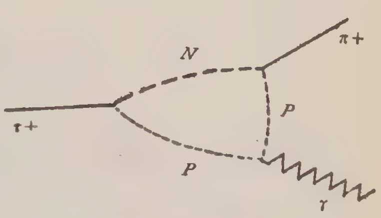
for the vector (pseudovector) coupling of scalar (pseudoscalar) meson.

$$H(p,\pi,\gamma) = -\left(\frac{cg}{\mu_\pi^2}\right)A_i(U_{i4}^*\varphi^+\tau_{Np}(\gamma_5)\gamma_4\varphi) - conj. \tag{3.8}$$

for the vector (pseudovector) coupling of vector (pseudovector) meson, and

$$H(p,\pi,\gamma) = -\left(\frac{cg}{2\mu_\pi^2}\right)\{[A, U^*]_{ij}\varphi^+\tau_{Np}\gamma_{ij} - 2[A, U^*]_{4j}\varphi^+\tau_{Np}\gamma_{4j}\varphi\} \tag{3.9}$$
$$-conj.$$

Process A:



for the tensor coupling of vector (pseudovector) meson. In (3.7) to (3.9) the normal to the variable surface is required by the integrability condition.[34]

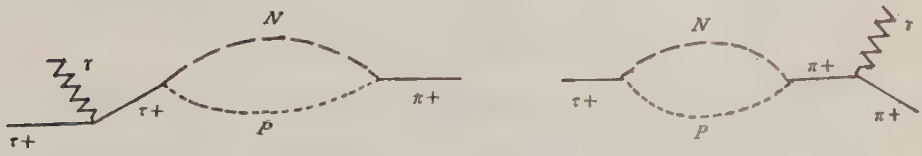
The decay process $\tau^+ \to \pi^+ + \gamma$ is visualized by Feynman's diagram[35] in Fig. 1. For each process the following contact transformation to eliminate virtual nucleon fields is carried out.

Process A.

$$-\int^{C} dx' \int^{C'} dx'' \{[H''(n,\pi), [H'(p,\gamma), H(n,\tau)]] + [H''(p,\gamma), [H'(n,\pi), H(n,\tau)]]\}$$
$$= \frac{1}{2} icgG \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx''\, U_M''^* A_i' V_L S_p \{\sum M\bar{S}(X''-X')\gamma_i\bar{S}(X'-X)LS^{(1)}(X-X'')\}, \tag{3.10}$$

where M and L are combinations of γ matrices as defined in Table III.

Process B.



35) R. P. Feynman, Phys. Rev. **76** (1949), 749, 769.
F. J. Dyson, Phys. Rev. **75** (1949), 426, 1756.

Process C.

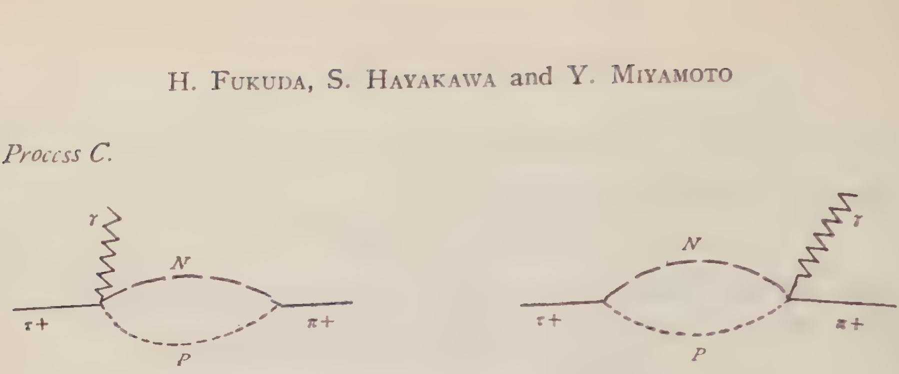


Fig. 1. Diagrams for $\tau^+ \to \pi^+ + \gamma$ decay.

Process B.

$$-\int^{C} dx' \int^{C'} dx'' \{[H''(\pi,\gamma)+H''(\tau,\gamma),\ [H'(n,\pi),\ H(n,\tau)]]$$
$$+[H'(n,\pi),\ [H'(\tau,\gamma),\ H(n,\tau)]]\}$$
$$=\frac{1}{2} icgG \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' A_i' \{\Lambda_{Mi}^{\pi}(X''-X')V_L + \Lambda_{Li}^{\tau}(X''-X)U_M'\}$$
$$\times S_p \{M\bar{S}(X'-X)LS^{(1)}(X-X') + MS^{(1)}(X'-X)L\bar{S}(X-X')\} \tag{3.11}$$

where

$$\Lambda_{Mi}^{\pi}(X''-X') = (i)[J_i'', U_M']$$
$$\Lambda_{Li}^{+\tau}(X''-X) = (i)[J_i'', V_L]. \tag{3.12}$$

$\Lambda_M^{\pi}(X''-X')$ and M are represented in Table III for respective types $\Lambda_L^{\tau}(X''-X)$ and L are the corresponding expressions for τ-meson and got from (3.12) and Table III by the substitution

$$X' \to X,\ U^* \to V,\ U_{ij}^* \to V_{ij}, \qquad \Lambda_{\pi} \to \Lambda_{\tau}.$$

Table III. Expressions for Λ and M.

Meson	Coupling	$\Lambda_M^{\pi}(X''-X')$	M
S (Ps)	s (ps)	$U''^*(\overrightarrow{\partial_i''}-\overleftarrow{\partial_i''})\Delta_{\pi}(X''-X')$	$1(\gamma_5)$
S (Ps)	v (pv)	$U''^*(\overrightarrow{\partial_i''}-\overleftarrow{\partial_i''})\partial_j'\bar{\Delta}_{\pi}(X''-X')$	$\gamma_j(\gamma_5\gamma_j)$
V (Pv)	v (pv)	$U_j''^*(\partial_i''\delta_{jk}-\partial_j''\delta_{ik})\bar{\Delta}_{\pi}(X''-X')$ $-U_{ij}''^*(\delta_{jk}-\frac{\partial_j''\partial_k''}{\mu_{\pi}^2})\bar{\Delta}_{\pi}(X''-X')$	$\gamma_k(\gamma_5\gamma_k)$
V (Pv)	t	$\partial_k'\{U_j''^*(\partial_i''\delta_{jl}-\partial_j''\delta_{il})\bar{\Delta}_{\pi}(X''-X')$ $-U_{ij}''^*\delta_{jl}\bar{\Delta}_{\pi}(X''-X')\}$	$\gamma_{kl}(\gamma_{kl}\gamma_5)$

We have only to take the contribution from the parts $\partial_j\partial_i\bar{\Delta}(X)$ in (3.12), making use of

$$-\frac{1}{2}\varepsilon(X)\partial_j\partial_i\Delta(X)=\partial_j\partial_i\bar{\Delta}(X)+\delta_{j4}\delta_{i4}\delta^4(X) \tag{3.13}$$

The second term in the right hand side of (3.13) cancels out with the part containing the normal direction in process C, which ensures the Lorentz invariance of whole matrix element.

Process C.

$$i\int^{C}dx'\,\{[H'(n,\pi,\gamma),\ H(n,\tau)]+[H'(n,\pi),\ H(n,\tau,\gamma)]\}. \tag{3.14}$$

Adding the contribution from the term $\delta_{j4}\delta_{i4}\delta^4(X)$ in process B, (3.14) results in for respective cases, (3.6), (3.7) and (3.8)

$$(3.6)\rightarrow(-iegG/2\mu_\pi)\int_{-\infty}^{\infty}dx'U'^*A_i'$$

$$\times S_p\{\gamma_5\gamma_i\bar{S}(X'-X)LS^{(1)}(X-X')+\gamma_5\gamma_iS^{(1)}(X'-X)L\bar{S}(X-X')V_L\} \tag{3.15}$$

$$(3.7)\rightarrow 0$$

$$(3.8)\rightarrow(-iegG/2\mu_\pi)\int_{-\infty}^{\infty}dx'A_i'U_j' \tag{3.16}$$

$$\times S_p\{\gamma_5\gamma_{ij}\bar{S}(X'-X)LS^{(1)}(X-X')+\gamma_5\gamma_{ij}S^{(1)}(X'-X)L\bar{S}(X-X')\}V_L \tag{3.17}$$

For the case of τ-meson the substitutions

$$X'\rightarrow X \quad LV_L\rightarrow MU_M'^*,\quad U_M'^*A_i'\rightarrow V_LA_i$$

are taken place in (3.14) to (3.16).

The contribution from processes B and C vanishes in almost all combination of the coupling of τ- and π-meson except

$$ss,\ vv,\ tt,\ pv\,pv,\ ps\,ps,\ vt,\ pv\,pv. \tag{3.18}$$

The combinations, $(s\,ps)$ and $(s\,pv)$, vanish by trace 0 and the other by Furry's theorem.

Gauge invariance. We must here see whether our formulation is gauge invariant or not. Applying the gauge transformation

$$A_i\rightarrow A_i+\partial_i\Lambda \tag{3.19}$$

to (3.10), and eliminating ∂_i' by integration by part, we get

$$(iegG/2)\int_{-\infty}^{\infty}dx'(\Lambda-\Lambda')U_M'^*V_L$$

$$S_p\{M\bar{S}(X'-X)LS^{(1)}(X-X')+MS^{(1)}(X'-X)L\bar{S}(X-X')\} \tag{3.20}$$

(3.20) vanishes if the contribution from processes B and C vanishes, since the integral is similar to those of B and C.

If it is not the case, the contribution from B and C results in the same and opposite sign as (3.20) by the gauge transformation, and there holds the gauge invariance with regard to whole matrix element $A+B+C$. As an example, we take pseudoscalar τ- and π-mesons with pseudoscalar couplings. In this case there is not process C. Accounting for (3.11) and Table III, process B results in

$$(icgG/2)\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''A_j''\{U''^*(\vec{\partial}_j''-\overleftarrow{\partial}_j'')\bar{\Delta}_\pi(X''-X')V-U''^*(\vec{\partial}_j''-\overleftarrow{\partial}_j'')\bar{\Delta}_\pi(X''-X)U'^*\}$$

$$\times S_p\{\gamma_5\bar{S}(X'-X)\gamma_5S^{(1)}(X-X')+\gamma_5S^{(1)}(X-X')\gamma_5\bar{S}(X-X')\}. \quad (3.21)$$

Carrying out the gauge transformation $A_j''\rightarrow A_j''+\partial_j''\Lambda''$ and by integration by part over X'', we get

$$(icgG/2)\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''\Lambda''\{\delta(X''-X)U''^*V-\delta(X''-X)U'^*V\}$$

$$\times S_p\{\gamma_5\bar{S}(X'-X)\gamma_5S^{(1)}(X-X')+\gamma_5S^{(1)}(X'-X)\gamma_5\bar{S}(X-X')\} \quad (3.22)$$

where we use a relation

$$\partial_j''\{U''^*(\vec{\partial}_j''-\overleftarrow{\partial}_j'')\bar{\Delta}_\pi(X''-X')\}=-U''^*\delta(X''-X').$$

Comparing (3.20) and (3.22), we see they cancel out in each other.

§ 2. The general rules for matrix.

1. *Furry's theorem.* As found in II § 2, the forbidden set takes place according as the coupling of π^0-meson with nucleon is 1 or τ_3. The corresponding coupling in this case, $\tau\rightarrow\pi+\gamma$, is τ_p, which is the sum of 1 and τ_3. Therefore, there exists no selection rule as in the case of $\tau\rightarrow\pi+\pi^0$ decay.

2. *Lorentz invariance of matrix element.* Among the forbidden condition in (2.7), only a set ($s\ v\ ps$) is applicable to this case, since the photon has vector character. This set is denoted by * in Table IV.

3. *Divergence theorem.* This theorem concerns with the scalar meson with the vector coupling. The $U_M''^*=\partial_iU''^*$ and $M=\gamma_i$ in (3.10). By integration by part we get

$$(icgG/2)\int_{-\infty}^{\infty}dx'(U''^*A_i'-U'^*A_i')V_L$$

$$\times S_p\{\gamma_i\bar{S}(X'-X)LS^{(1)}(X-X')+\gamma_iS^{(1)}(X'-X)L\bar{S}(X-X')\}. \quad (3.23)$$

The part containing U'^*A_i' cancels out with the process C. The other part vanishes except for $L=\gamma_i$ and γ_{ij} by Furry's theorem. But we can put this zero when $L=\gamma_i$, since this is the same type as the self-energy of photon. In

the case of $L=\gamma_{ij}$, there remains only the first term of expansion, by making use of $\Box A=0$. If the coupling $\gamma_{ij}(\gamma_{ij}\gamma_5)$ is the case of vector or pseudovector meson, the result is expressed as

$$-\frac{\varkappa}{4\pi}\frac{eg}{\mu_\pi\mu_\tau}\frac{1}{2}V_{ij}F_{ij}U^*(\log\infty),$$

which is divergent but gauge invariant. This can be eliminated by the regulator. The above procedure holds by the interchange of π- and τ-mesons.

As for the contribution from process B, it is shown that (3.11) vanishes by integration by part.

Thus the vector coupling of scalar meson is entirely forbidden as denoted by D in Table IV.

4. *Equivalence theorem.* Putting $U_M^*=\partial_i U^*$ and $M=\gamma_5\gamma_i$, (3.10) results in

$$(\text{equivalent term})+(iegG/2\mu_\pi)\sum\int_{-\infty}^{\infty}dx' A_i' V_L$$

$$\times\{U'^* S_p \sum \gamma_5\gamma_i \bar{S}(X'-X)LS^{(1)}(X-X')+U^* S_p\sum\gamma_i\bar{S}(X'-X)L\gamma_5 S^{(1)}(X-X')\} \tag{3.24}$$

by integration by part. The first term in the second part cancels out with (3.15). The second term vanishes except for the τ-meson with the tensor coupling. The contribution from process B, (3.11), is found to hold the equivalence theorem. Thus the equivalence is ensured except for the partner meson with tensor coupling.

5. *The forbiddenness of* (SS) *and* (P_sP_s). Assuming the gauge invariance, the interaction Hamiltonian is represented by the linear combination of $F_{ij}\partial_i U^*\partial_j V$, $F_{ij}(\partial_i\partial_j U^*)V$ and $F_{ij}(\partial_i\partial_j V)U^*$. Remarking that F_{ij} and $\partial_i\partial_j$ are the antisymmetrical and symmetrical tensor, respectively, these terms are found to vanish. Also by the direct calculation, we get the vanishing result except a non-gauge invariant term which should be dropped. These forbidden sets are denoted by G in Table IV.

§ 3. Results.

We calculate the life times for allowed sets, in which the masses and coupling constants are adopted as in (2.27) and (2.28). The divergences are dropped by regulators

$$\int dx\rho(x)=0,\qquad \int dxx\rho(x)=0,\qquad \int dxx^2\rho(x)=0,$$

$$\int dx\log|x|\rho(x)=0,\quad \int dxx\log|x|\rho(x)=0,\quad \int dxx^2\log|x|\rho(x)=0. \tag{3.25}$$

Since there is no experimental evidence for the regulator but some inconsistence as mentioned before, the life times represented in Table IV give only the upper limits of real values. This procedure gives yet too short life times to explain the experiments.

Table IV. Life times of $\tau \rightarrow \pi + \gamma$ decay.

τ(S, s)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency			con.	div.	con.	div.		
Life	*G*	*D*	$8.0\times10^{-[illegible]}$	2.8×10^{-13}	3.2×10^{-14}	1.8×10^{-13}	*	*

τ(S, v)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency				div.		div.		
Life	*DG*	*DS*	*D*	R_1	*D*	R_1	**D*	**D*

τ(V, v)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency	con.		div.	con.	con.	div.	con.	con.
Life	2.4×10^{-14}	*DS*	$3.8\times10^{-[illegible]}$	$6.4\times10^{-[illegible]}$	1.5×10^{-13}	1.5×10^{-15}	1.4×10^{-13}	1.1×10^{-14}

τ(V, t)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency	div.		con.	con.	con.	div.	con.	div.
Life	7.6×10^{-12}	R_1	6.4×10^{-14}	3.2×10^{-13}	3.7×10^{-15}	6.0×10^{-15}	3.8×10^{-15}	5.5×10^{-13}

τ(Pv, pv)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency	con.		con.	con.	div.	con.	div.	div.
Life	2.5×10^{-14}	*D*	1.2×10^{-11}	1.2×10^{-14}	9.0×10^{-12}	2.5×10^{-14}	3.6×10^{-12}	9.5×10^{-14}

τ(Pv, t)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency	div.		div.	div.	con.	con.	con.	div.
Life	1.9×10^{-12}	R_1	1.2×10^{-13}	6×10^{-14}	2.5×10^{-14}	1.9×10^{-11}	$2.4\times10^{-[illegible]}$	3.4×10^{-12}

τ(Ps, pv)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency			con.	con.	div.	con.		
Life	*	D	4.1×10^{-13}	1.3×10^{-15}	5.1×10^{-14}	1.5×10^{-15}	G	G

τ(Ps, ps)

π	S		V		Pv		Ps	
Coupling	s	v	v	t	pv	t	pv	ps
Divergency			con.	div.	div.	div.		
Life	*	D	3.5×10^{-15}	2.0×10^{-14}	3.9×10^{-16}	2.3×10^{-14}	G	G

Note:
*: Forbidden because of no Lorentz invariant matrix.
G: Forbidden by gauge invariance.
D: Forbidden by divergence theorem.
S: The same form as the self-energy of photon.
R_1: Forbidden by regulator.

From the above we see there are few cases to be forbidden the decay modes $\tau\rightarrow\pi+\pi^0$ and $\tau\rightarrow\pi+\gamma$. As for $\tau\rightarrow\pi+\pi^0$ decay, we shall adopt symmetrical theory, which may be preferable considering the nuclear force. And we may assume the $\pi^{\pm}$ and π^0 mesons belong to the same type with the same coupling. Then the acceptable cases for the sets of τ- and π-mesons are shown in Table V, with the sets having the longer life times than 10^{-11} sec.

Considering the fact only that the neutral meson may disintegrate into two photons[7], however, the scalar or pseudoscalar π-meson is acceptable. Among both possibilities regarding with π-meson, we may adopt the pseudo type, because the scalar meson gives entirely wrong nuclear force. It is of interest that the pseudoscalar π-meson is preferable in our case, too, because the decay of τ-meson with ps and (Ss) is completely forbidden for two-body decay. This problem will be clarified by considering the $\tau\rightarrow3\pi$ decay, which is experimentally found.

IV. The Decay of $\tau\rightarrow3\pi$.[36]

The decay mode, $\tau\rightarrow3\pi$, may be the most probable one, if $\tau\rightarrow\pi+\pi^0$ and $\tau\rightarrow\pi+\gamma$ are forbidden or considerably slow. These cases are shown in Table V and treated below.

The same attempt has been carried out by Nakamura in perturbation

36) H. Fukuda, Soryushi-Ron Kenkyu, Vol. I, No. 3, (1949), 1.

method[37] and by Power in Feynman's method.[38] They give the same order of life times as ours. Nevertheless, the application of perturbation method to this problem is somewhat questionable as was the case of Tanikawa and Finkelstein in γ-decay problem.[39] And the covariant formulation may be necessary. Power's calculation is based on the covariant formalism, but she treats only a case that all mesons are pseudoscalar type with pseudoscalar coupling, though this set is most plausible. Our calculation for the possible types in Table V shows that life times are considerably dependent on the types and couplings of these mesons. Then there may be possibly the third order decay unless the second order is forbidden.

Table V.
Allowed sets for $\tau\rightarrow\pi+\pi^0$ or γ decay.

	τ	π
1	(S, v)	S, Pv
2	S, Pv, Ps	(S, v)
3	S Ps	S Ps
4	(S, s) Ps	Ps S
5	(Pv, pv)	(V, v)
6	(Pv, pv)	(Pv, pv)

About a half of matrix elements under consideration are divergent and not free from ambiguity. Such divergence and ambiguity may be dropped by the regulator condition

$$\int \rho(x)dx=0, \quad \int dx \log|x|\ \rho(x)=0. \tag{4.1}$$

The life times obtained by this prescription are considered to give the upper limit of real ones.* (This procedure may not be justice according to the problem. The self-energy of photon may belong to the former case, but the logarithmic divergence as appeared in the self-energy of electron may be the latter case.)

§ 1. Fundamental formulation.

The calculation is similar to the case of the $\tau\rightarrow\pi+\pi^0$ decay. Only change is the number of π-mesons, three instead of two. Their wave functions or their derivatives are denoted by U_L, U_M and U_N, corresponding to positive (negative), negative (positive) and positive (negative) π-mesons, respectively. The corresponding one for τ-meson is denoted by U_τ. After this manner the suffixes are changed from II. Then the fundamental expression of the matrix element is given by

$$M=\frac{1+(-1)^N}{4}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''\int_{-\infty}^{\infty}dx'''\ U_L'U_M''U_N'''U_\tau$$

$$S_p\{\sum \bar{S}(X-X')\gamma_L\bar{S}(X'-X'')\gamma_M\bar{S}(X''-X''')\gamma_N S^{(1)}(X'''-X)\gamma_\tau\} \tag{4.2}$$

corresponding to (2.4′). Here N means the number of γ_i and γ_{ij} in $\gamma_L, \gamma_M, \gamma_N$ and γ_τ.

37) S. Nakamura, unpublished.
38) S. Power, Phys. Rev. **76** (1949), 865.
39) Y. Tanikawa, Prog. Theor. Phys. **3** (1948), 38.
R. J. Finkelstein, Phys. Rev. **72** (1947), 414.

* In the higher order than the third there is no ambiguity and is not necessary to use regulators.

Further we make the reasonable assumption that three π-mesons are the same type. The other decay mode, $\tau\rightarrow\pi+\tau^0$, $\tau^0\rightarrow 2\pi$, as suggested by Ozaki, is not plausible, since the two fast mesons are both positive in the Bristol decay.[11]

§ 2. General rules for matrix.

1. *Furry's theorem.* From the factor $1+(-1)^N$ there are the allowed cases that the couplings of both π- and τ-mesons belong to either the class of scalar, pseudovector and pseudoscalar or the class of vector and tensor, since three π-mesons have the same coupling. Then the possibility 5 in Table is ruled out.

2. *Lorentz invariance.* The forbidden cases are the set of scalar (pseudoscalar) τ-meson and pseudoscalar (scalar) π-mesons. Then the case 4 is ruled out.

3. *Divergence theorem.* In the present case the forbidden does not take place by this theorm.

4. *Equivalence theorem.* This theorem does not hold throughout in the present case.

5. *Symmetry character.* The matrix element has symmetry with respect to two π-mesons.

§ 3. Evaluation of matrix element.

The procedure is similar to II and here is given only an outline. Transforming (4.2) into momentum space we obtain

$$M=\frac{1}{2(2\pi)^3}\int dk\int dk'\int dk''\int dk'''\delta(k-k'-P_L)\delta(k'-k''-P_M)\delta(k''-k'''-P_N)$$
$$S_p\{i\gamma k-\varkappa)\gamma_L(i\gamma k'-\varkappa)\gamma_M(i\gamma k''-\varkappa)\gamma_N(i\gamma k'''-\varkappa)\gamma_\tau\}$$
$$\times U_L U_M U_N U_\tau \sum_{cyc.}\frac{\delta(k'''^2+\varkappa^2)}{(k^2+\varkappa^2)(k'^2+\varkappa^2)(k''^2+\varkappa^2)} \tag{4.3}$$

where P_L, P_M, P_N and P_τ mean the momenta of three π-mesons and a τ-meson.

For example, in the case of pseudoscalar π- and τ-mesons with pseudoscalar coupling (4.3) is reduced to

$$M=\frac{1}{2(2\pi)^3}\int dk\int dk'\int dk''\int dk'''\delta(k-k'-P_L)\delta(k'-k''-P_M)\delta(k''-k'''-P_N)$$
$$\times U_L U_M U_N U_\tau\Big[2\int_{-1}^{1}da_1(-\Gamma_1)\{\delta'(\varkappa^2+\xi k^2+\xi_1 k''^2)+\delta'(\varkappa^2+\xi k'^2+\xi_1 k'''^2)\}$$
$$+2\sum_{cyc.}^{(LMN\tau)}(P_M\cdot P_N)\int_{-1}^{1}db_1\int_{-1}^{1}db_2\Gamma_2\delta''(\varkappa^2+\eta k^2+\eta_1 k'^2+\eta_2 k''^2)$$
$$+\{P_\tau^2\cdot P_N^2+P_\tau^2\cdot P_M^2-(P_L+P_M)^2(P_M+P_N)^2\}$$
$$\times\Big\{\int_{-1}^{1}dc_1\int_{-1}^{1}dc_2\int_{-1}^{1}dc_3(-\Gamma_3)\delta'''(\varkappa^2+\zeta k^2+\zeta_1 k'^2+\zeta_2 k''^2+\zeta_3 k'''^2)\Big\}\Big], \tag{4.4}$$

here

$$\xi=\frac{1+a_1}{2},\ \xi_1=\frac{1-a_1}{2};\ \eta=\frac{1+b_1}{2},\ \eta_1=\frac{1-b_1}{2}\frac{1+b_2}{2},\ \eta_2=\frac{1-b_1}{2}\frac{1-b_2}{2};$$

$$\zeta=\frac{1+c_1}{2},\ \zeta_1=\frac{1-c_1}{2}\frac{1+c_2}{2},\ \zeta_2=\frac{1-c_1}{2}\frac{1-c_2}{2}\frac{1+c_3}{2},\ \zeta_3=\frac{1-c_1}{2}\frac{1-c_2}{2}\frac{1-c_3}{2} \tag{4.5}$$

$$\Gamma_1=\frac{\xi\cdot\xi_1}{\frac{1-a_1^2}{2}}=\frac{1}{2},\ \Gamma_2=\frac{\eta\cdot\eta_1\cdot\eta_2}{\frac{1-b_1^2}{2}\ \frac{1-b_2^2}{2}},\ \Gamma_3=\frac{\zeta\cdot\zeta_1\cdot\zeta_2\cdot\zeta_3}{\frac{1-c_1^2}{2}\ \frac{1-c_2^2}{2}\ \frac{1-c_3^2}{2}}. \tag{4.6}$$

The method of calculation is so complicated that the detail is not presented here. Some formulas are shown in Appendix *B*.

§ 4. Results.

The life time is calculated as

$$\tau_0^{-1}=\frac{1}{4\pi^3}g_1^2g_2^2g_3^2G^2\mu_\tau\int_1^{\frac{a^2-3}{2a}}d\varepsilon\int_{\frac{a-\varepsilon-\Delta}{2}}^{\frac{a-\varepsilon+\Delta}{2}}d\varepsilon'\sum_{\text{spin}}|M|^2 \tag{4.7}$$

where

$$\varepsilon=E_L/\mu_\pi,\quad \varepsilon'=E_N/\mu_\pi$$

mean the energies of π-meson divided by the mass, and

$$a=\mu_\tau/\mu_\pi,\quad \Delta=\sqrt{\frac{(a^2-2a\varepsilon-3)(\varepsilon^2-1)}{a^2-2a\varepsilon+1}}$$

a is the ratio of τ- to π-meson masses.

By assuming the values of masses* and coupling constants as before, we get the life times for the possible sets as seen in Table IV. The dependency of the life times on the masses is represented by

$$\tau_0\propto\frac{1}{\mu_\pi}\left(\frac{\mu_\tau}{\mu_\pi}\right)^\lambda\Delta^{-\nu}F(\mu_\pi/\mu_\tau) \tag{4.8}$$

where $F(\mu_\pi/\mu_\tau)$ is the complicated function and slowly increases with decreasing μ_π/μ_τ but does not appreciably affect the life times as far as $\mu_\tau=950\sim1050$ m. And

$$\Delta=(\mu_\tau-3\mu_\pi)/\mu_\pi. \tag{4.9}$$

This factor largely affects the result. The rather long time obtained by Power may be due to the smaller values, $\mu_\tau\sim900$ m and $\mu_\pi\sim285$ m, against the experiment. λ is integer 2 or 4, according to the convergent or divergent case.

* $\mu_\tau=1000$ me $\quad\mu_\pi=300$ me

The former case takes place when the odd number of γ_i are contained in γ_L, γ_M, γ_N and γ_τ, while the latter case corresponds to that of the even number of γ_i, where we must use the regulator to eliminate the divergence. ν is also integer and represented in Table VI.

Table VI. The lifetimes for $\tau \rightarrow 3\pi$ Decay.

Type of	S	S	S	S	S	Pv	Ps	Ps	Ps	Ps
Coupling of	s	v	v	v	v	t	ps	ps	pv	pv
Type of	S	S	V	V	Pv	S	Ps	Ps	Ps	Ps
Coupling of	s	v	v	t	t	v	ps	pv	ps	pv
Divergency.	∞	∞	∞	C	C	C	∞	C	C	∞
Lifetime in sec.	$6 \cdot 10^{-10}$	$1.8 \cdot 10^{-9}$	$6 \cdot 10^{-11}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$1.4 \cdot 10^{-8}$	$1.9 \cdot 10^{-11}$	$1.8 \cdot 10^{-13}$	$1.8 \cdot 10^{-12}$	$1.8 \cdot 10^{-11}$
ν	2	4	3	3	4	6	2	2	2	2

If neutral theory is adopted, the following sets must be taken instead of the first and fifth sets above.

Type of	V	V
Coupling of	v	t
Type of	S	S
Coupling of	v	v
Divergency.	∞	C
Life times in sec.	2.9×10^{-11}	2.0×10^{-12}
	3	3

∞: Divergent, dropped by regulator.

C: Convergent.

The life time for the $\tau \rightarrow 3\pi$ decay may be said to give the right order consistent with experiments, if we take into consideration the ambiguous values of coupling constants. It may be better that the values of G and F are determined on the basis of this calculation. The most plausible set, *ps* and *ps*, gives the right life times for $f^2 \sim 10^{-1}$ and $F^2 \sim 10^{-3}$, but the set in relation with pseudovector coupling gives somewhat shorter lives. This suggests that g and G are smaller than f and F reconciled with the nuclear force. But here one must remark that the life times depend on f^6 or g^6, and then the slight change in the value f or g gives rise to the large change in the life time. Thus the precise determination of these values is much important for our problem. The more detailed discussions will be given in the next part.

V. Discussions.

We may draw some conclusions about meson problems from the above, although there is the inevitable ambiguity of the quantum field theory left untouched. The settlement of such difficulty will be postponed in future development and now we want to pursue in what extent our tentative prescription is

justified in comparison with experiments or what inconsistency gives rise in the interpretation of experiments.

§ 1. The types of mesons.

We see in II to IV there are few cases in which a τ-meson is possible to disintegrate into three π-mesons with the suitable life time. Even in the possible cases, as shown in Table VI, the sets except Ps and Ps may be ruled out, provided that each meson has two kinds of couplings. We can give several evidences for pseudoscalar π-meson in what follows.

(1) *γ-decay of neutral meson.* See, Introduction, references 6 and 7. In this evidence the possibility of scalar type is not ruled out.

(2) *The decay of τ-meson*, provided τ-meson is pseudoscalar. Other types are not necessarily ruled out if the rather complicated combination of special couplings are taken.

(3) *The angular distribution of produced π-meson* by γ-rays in Berkeley experiments. $\sin^2\theta$ distribution observed can be interpreted only by assuming the π-meson as scalar or pseudoscalar. Taking into consideration the effect of the recoil of nucleon, however, there appears considerably discrepancy from $\sin^2\theta$ distribution in the pseudoscalar case.[40] Scalar meson gives better fit concerning this point. This discrepancy may or may not vanish by considering the higher order effect or the bounding of the nucleon in the nucleus.*

The first evidence is concerned with neutral π-meson and the third is charged one.

The second is related to charged and neutral mesons but less certain. These evidences are favourable to scalar type, too, but it may be unacceptable if π-meson responds to the main part of nuclear force. Nevertheless, the nuclear force seems to be not explained by such a simple type of π-meson. It may be due to either the wrong mathematical treatment of the erroneous notion to ascribe the nuclear force to π-meson alone or both. Apart from such deep difficulty, pseudoscalar meson is considered to have advantage to other three types of meson with regard to the treatment of deuteron system.

Several arguments as mentioned above would imply how strong the interaction between a nucleon and a π-meson, if it should be pseudoscalar. The magnitude of its coupling constant, $f^2\sim40$, currently adopted, is supposed to be so great that the expansion with regard to the power of f may be unacceptable. Whenever it may be smaller, $f^2\sim4$, considering the fourth order process,[41] the perturbation theoretical treatment will still be doubtful. Nevertheless, the following

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40) Z. Kobu, M. Kotani and S. Nakai, Prog. Theor. Phys. in press K. A. Brueckner and M. L. Goldberger, unpublished.

41) H. A. Bethe, Phys. Rev. 76 (1949), 191. K. M. Watson and J. V. Lepore, Phys. Rev. 76 (1949), 115.

* See note added in proof.

phenomena directly related to π-meson seem to make us infer the weaker interaction.

(1) *$\tau\rightarrow 3\pi$ decay.* The life time for this process is given in the case of pseudoscalar couplings, only if we assume $f^2\sim 10^{-1}$ and $F^2\sim 10^{-3}$. The latter figure has much ambiguity, but is not so smaller than 10^{-4} considering the abundance of τ-mesons as discussed in I. Then the magnitude of f^2 is impossible to exceed two or three times of this figure because the life depends on f^{-6}. If the value of f^2 is thus settled, τ-meson can rapidly disintegrate through the pseudovector couplings, provided $g^2\sim 10^{-1}$ The magnitude of f^2 should, however, be larger than g^2 if we take into consideration the equivalence between pseudoscalar and pseudovector couplings for the nuclear force According to the equivalence condition, f^2 is $(2\varkappa/\mu)^2$ times g^2, which results too shorter lifetime for (ps, ps) set. As for the pseudovector coupling of meson, we have so little knowledge as to specify its magnitude, but it is unlikely to take the equivalence theorem.

(2) *The production of π-mesons by γ-rays.** Comparing the calculated cross section[40] for this process with an experiment, f is considered to lie between 10^{-1} and 1. This argument seems to suggest the smaller magnitude of f^2, though the calculation is not yet so complete that pseudovector coupling, the higher order process and the effect of binding in a nucleus must be considered.

(3) *The production of π-mesons by nucleon-nucleon collision.* The cross section calculated meson theoretically also suggests smaller f^2 than 1 but somewhat larger than 10^{-1}.[42), 43)] But in this case the production may undergo the effect of the more complicated nature of nuclear force, which obscures the determination of f^2 value.[44)] To see this the more investigation is reguired concerning what values of f^2 and g^2 must be taken when the nuclear potential is replaced by mesonic interaction.

The nuclear potential adopted by Serber, as referred in many literatures, from the analysis of the proton-neutron scattering for 90 *MeV* is $\sim 0.3 \exp(-\lambda r)/r$. The factor 0.3 should be brought about from $\sim f^4 v^2$, according to psoudoscalar meson theory. This implies f^2 as the order of unity. But the cross section reciprocally propoitional to incident energy suggests that the higher order effect must be accounted for. We are again led to the presumption that the nuclear force would be very complicated phenomena and, perhaps, could not be explained by π-meson.

Further, there is one more evidence for the weak interaction of π-meson with nucleon. The fraction of transferred energy to mesons by a collision with

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42) C. Morette and H. W. Peng. Proc Roy. Ir. Acad. **51** (1948), 217.
S. Takagi, Prog. Theor. Phys. **4** (1949) 557.
C. Morette, Phys Rev. in press (only pseudoscalar coupling).

43) C. Richman and H. A. Wilcox, unpublished. M. Weisskopf, unpublished.

44) c. f. L. L. Foldy and R. E. Marshak, Phys Rev. **75** (1949) 1493.

* This argument is not true for error of calculation. The corrected value of coupling constants from this experiments are $f^2\sim 17$ and $g^2=0.1$ (see note added in proof.)

an air nucleus is estimated as about 1/5.[45] Thus only one tenth of the incident energy is imparted to π-mesons by a nucleon-nucleon collision with the energy about 10 *Bev*. This may also be supported by the analysis of the meson production, as will be discussed in a separate paper.

§ 2. Varieties of mesons.

There have been few examples for the existence of mesons other than π and μ meson. Among them the existence and the nature of τ-meson are considered to be almost established as discussed above. Besides there seems to be another τ-meson discovered by Wagner and Cooper[32] which resembles much to τ-meson and is often confused with it. Its mass, 725 ± 40 m, may not only allow to decay into three π-mesons, but also a star produced by one of those mesons has so small number of prongs that its rest energy is not effectively utilized.[46] The more evidences seem to be obtained by Armenian authors, who observed some mesons with mass about 700 m, starting from stars and ending in the emulsion without secondary track.[18] The considerably more frequent appearance of these mesons than τ-mesons suggests that this sort of mesons, tentatively called as τ'-meson,[46] may appreciably responds to nuclear force. Detailed analysis concerning τ'-meson will be published in a separate paper.

The cloud chamber evidence for the decay of heavier mesons[47)–50] are usually considered to be due to τ-mesons.* The estimation of their masses are based on that they disintegrate into two mesons. If this interpretation were correct, there would be the decay mode, $\tau \rightarrow \pi + \pi^0$ or γ. Then we can hardly obtain any appropriate set with the life time of two particle decay as long as three particle decay, unless any new condition to lower the decay probability be added. We may not now consider this kind of mesons are τ-mesons, though there is a possible explanation like as to consider the mass difference of proton and neutron,[9] since they seem to have somewhat long lives and may be supposed as a kind of varitrons. The life time of this kind of mesons is estimated from Rochester and Butler's data. They estimated the momentum of a parent heavier meson as about 0.6 *Bev/c*. The path length of this meson may be about 15 cm, since it seems to be produced in the adjacent material and disintegrate near the center of the cloud chamber with the diameter 30 cm. Then the life time is the order of 5×10^{-10} sec., which is much longer than that of τ-meson estimated in I. Two these mesons are observed in 50 penetrating showers at sea level. It is

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45) S. Hayakawa and J. Nishimura. J. Sci. Res. Inst. **44** (1949), 47. Prog. Theor. Phys. **4** (1948), 232, 577.

46) S. Hayakawa and Y. Yamaguchi, Prog. Theor. Phys. **4** (1949), 570.

47) J. Dandin. Ann. de Phys. **19** (1944), 110.

* After this discovery Tanikawa proposed four meson hypothesis on the aesthetical point of view, in which there are the interaction only between Fermions and Bosons. Prog. Theor. Phys. **3** (1948), 314, 315.

supposed to be considerably frequent.

Many varieties of mesons appeared as frequent as above mesons and with longer lives are discovered by Armenian authors by making use of a powerful magnetic analyzer.[48] The same kinds of mesons hardly identified with μ or π-meson are also discovered by cloud chambers.[49] They should have considerably long life times in order to pass through their apparatuses without disintegrations. A greater part of them may be produced by the disintegrations of stopped varitrons accounting for the existence of maximum momenta in the momentum spectrum.[48] Most of them are likely to decay with lives as long as 10^{-6} sec.[50] Furthermore there are few cases of nuclear disintegrations at their range ends.[49),51),18] Above facts make us infer that most of varitrons do not directly interact with nucleons.[52]

If they were nuclear mesons, they could not be stable as found in II to IV. Generally speaking, a mixed field theory is very difficult to be reconciled with the decay problem. For example, the hypothetical C-meson[53] would break the forbidden of $\tau \rightarrow \pi + \gamma$ decay, unless its mass is greater than ~ 600 m. Thus they can hardly be nuclear mesons but may be the descendants of nuclear mesons, accounting for that they are likely produced by nuclear interactions.[54] The strongest argument for this view will be the existence of lighter varitrons than π-mesons, as was suggested by Taketani.[55] If they were nuclear mesons, they sould be observed in Berkeley experiments, in which the threshold energy for the production of these mesons is low enough and the intensity of produced mesons is so great that the weak interacting mesons will also be observed. Accordingly, they are supposed to be the decay products of heavier nuclear mesons. We are, therefore, led to the view that varitrons are cousins of μ-mesons or their descendants. How many sorts of varitrons really exist and what is their nature will be clarified by the future development. In that time our work, though appearing superfluous in the present day, will give an useful tool to discriminate the complicated varieties.

§ 3. Concluding remarks.

From the above analysis we can partly answer the problem proposed in Introduction. The application of the quantum field theory to meson problems may

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48) A. I. Alikhanian, A. I. Alikhanov and A. O. Vaisenberg, J. Exp. Theor. Phys. **18** (1948), 301. A. I. Alikhanian. V. M. Morozov and A. V. Khrimian, C. R. Acad. Sci. U. S. S. R. **61** (1948), 39. We cite here only available papers.

49) J. G. Retallack and R. B. Brode, Phys. Rev. **75** (1949), 1716. R. B. Brode. Rev. Mod. Phys. **21**, (1949), 37.

50) A. O. Vaisenberg, J. Exp. Theor. Phys. **10** (1949), 726.

51) A. I. Alikhanian, D. M. Samoilovich, I. I. Gurevich. Kh. P. Balayan and R. I. Gerasimove, J. Exp. Theor. Phys. **19** (1949), 1716.

52) S. Hayakawa, Prog. Theor. Phys. in press. Kagaku, **20** (1950), 52.

53) A. Pais, Phys. Rev. **68** (1945), 227; S. Sakata, Prog. Theor. Phys. **2** (1947), 145, 73.

54) A. I. Alikhanian, M. I. Daison and V. M. Kharumonov, J. Exp. Theor. Phys. **19** (1949), 739.

55) M. Taketani, Private conversation.

give the considerably satisfactory results capable to explain some experiments. Nevertheless, we cannot help leaning the conventional prescription to remedy the ambiguity accompanied by our treatment. Such prescription is supposed to be not at all untrustworthy and suggests a way to the true theory. One must, however, not forget that it is, at any rate, a conventional one and can not be used without criticism. Then, the life times adopted by us only give the upper limits of true values. It seems to be a hopeful fact that the life times are not inconsistent with experiments and we expect that the quantum field theory will still be valid in future as a good correspondent theory. Furthermore, the model of mesons, which seem to unexpectly weakly interact with nucleons, is again close to the truth, but it is doubtious that π-meson bears practically all part of nuclear forces. Thus our study is not the solution but the presentation of questions.

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We should like to express our heartiest thanks to Professors Tomonaga and Yukawa who have often communicated us the foreign works and encouraged us. We are also obliged to Dr. Taketani for his invaluable suggestion, especially various modes of production mechanisms of τ-meson, and to Dr. Miyazima for his advices in the use of regulator and others. The deepest gratitude is offered to Dr. Ozaki and his colleagues. We could not promote our work without the frequent discussion with them in the course of our calculation. Dr. Steinberger has also attempted the similar work and contributes to our work by sending his paper before publication. Mr. Nakamura has always given us the invaluable advices and guided us by his unpublished result about $\tau \rightarrow 3\pi$ decay. We also thanks to Messrs. Koba, Nambu, Fujimoto and Yamaguchi for their stimulating discussions.

Appendix.

We treat the decay of a τ-meson into n lighter Bose mesons, as it is of formally interest.

According to Tomonaga-Schwinger theory the decay of a meson into n lighter mesons are described by the effective Hamiltonian density

$$H_I=\int_{-\infty}^{\infty} dX' \cdots dX^{(n)} \sum_{i=0}^{n} S_p \{ \bar{S}(X-X')\gamma_L S(X'-X'')\gamma_M \cdots \gamma_P S^{(1)}(X^{(i)}-X^{(i+1)})\gamma_Q \cdots$$

$$\cdots \gamma_R \bar{S}(X^{(n-1)}-X^{(n)})\gamma_S \bar{S}(X^{(n)}-X)\gamma_T \} \tau_L \tau_M \cdots \tau_S \tau_T U'_L U''_M \cdots U^{(n)}_S U_T \qquad \text{(A1)}$$

since there is no nucleon before and after the interaction and we are not interested in the radiative correction. (in this approximation $U'_L U''_M \cdots U_T$ may be assumed to commute with each other.) Here γ_L *etc.* represent 1, γ_i, γ_{ij}, $\gamma_5\gamma_i$ and γ_5 according as U_L *etc.* are scalar, vector, tensor, pseudovector and pseudoscalar respectively, τ_L *etc.* are either 1 or τ_3 when U_L *etc.* belong to the neutral meson field, and

τ_{PN} or τ_{NP} if U_L *etc.* describe the charged meson. U_L *etc.* are potentials (including coupling constant) or their derivatives describing meson fields.

A) *Selection rule.*

We show the selection rule for the matrix element. Corresponding to the process described by (A1), there is the process described by

$$H_{II}=\int_{-\infty}^{\infty} dx'\cdots dx^{(n)}\sum_{i=0}^{n} S_p\{\gamma_T\bar{S}(X-X^{(n)})\gamma_S\bar{S}(X^{(n)}-X^{(n-1)})\gamma_R\cdots\gamma_Q S^{(1)}(X^{(i+1)}-X^{(i)})\gamma_P$$
$$\cdots\gamma_M\bar{S}(X''-X')\gamma_L\bar{S}(X'-X)\}\tau_T\tau_S\cdots\tau_M\tau_L U_L' U_M''\cdots U_S^{(n)}U_T. \tag{A2}$$

We first concern only the case where not all of the meson under consideration are neutral. Then it follows that

$$\tau_L\tau_M\cdots\tau_T\tau_S=(-1)^{N_3}\tau_T\tau_S\cdots\tau_M\tau_L \tag{A3}$$

(N_3 denotes the number of τ_3 spin occurring in this process.) accounting for

$$\tau_{PN}\tau_3=-\tau_{PN},\quad \tau_{NP}\tau_3=\tau_{NP},\quad \tau_3^2=1.$$

We notice further

$$S_p(\gamma_i\gamma_j\cdots\gamma_p\gamma_q)=S_p(\gamma_q\gamma_p\cdots\gamma_j\gamma_i).$$

Applying this relation to the trace of (A2) we obtain the trace part of (A2).

$$S_p\{\bar{S}^*(X-X')\gamma_L^*\bar{S}^*(X'-X'')\gamma_M^*\cdots\gamma_P^*S^{(1)*}(X^{(i)}-X^{(i+1)})\gamma_Q^*\cdots$$
$$\cdots\gamma_R^*\bar{S}^*(X^{(n-1)}-X^{(n)})\gamma_S^*\bar{S}^*(X^{(n)}-X)\gamma_T^*\} \tag{A4}$$

where $\gamma_R^*=\varepsilon\gamma_L$, $\varepsilon=1$ for $\gamma_L=1$, γ_i, γ_5 and $\varepsilon=-1$ for $\gamma_L=\gamma_{ij}$, γ_{ijk}. Thus (A4) turns into

$$S_p\{\bar{S}^*(X-X')\gamma_L\bar{S}^*(X'-X'')\gamma_M\cdots\gamma_R\bar{S}^*(X^{(n-1)}-X^{(n)})\gamma_S$$
$$\times\bar{S}(X^{(n)}-X)\gamma_T\}\times(-1)^{N_t+N_{pv}} \tag{A4$'$}$$

(N_t, N_{pv} denote the number of tensor and pseudovector couplings in this process.) Remembering the fact that the trace of the product containing odd number of γ's vanishes and that

$$\bar{S}(X)=\left(\gamma\frac{\partial}{\partial X}-\varkappa\right)\bar{\Delta}(X),\qquad \bar{S}^*(X)=\left(-\gamma\frac{\partial}{\partial X}-\varkappa\right)\bar{\Delta}(X)$$

$$S^{(1)}(X)=\left(\gamma\frac{\partial}{\partial X}-\varkappa\right)\Delta^{(1)}(X),\quad S^{(1)*}(X)=\left(-\gamma\frac{\partial}{\partial X}-\varkappa\right)\Delta^{(1)}(X)$$

the trace part of (A4′) is reduced to

$$\text{Spur part of (A.1)}\times(-1)^{N_v+N_{pv}} \tag{A5}$$

(N_v denotes the number of vector couplings occurring in this process.) Combining (A3), (A4′) and (A5), we obtain finally

$$H_{II}=H_I\times(-1)^{N_v+N_t+N_3}. \tag{A6}$$

We see from (A6) that the matrixelement (A1) and (A2) cancel out with each other if $N_\nu + N_\iota + N_3$ equals an odd number, so that such a process is forbidden.

In the case where all mesons are neutral, we obtain

$$\tau_L \tau_M \cdots \tau_S \tau_T = \tau_T \tau_S \cdots \tau_M \tau_L \tag{A3'}$$

instead of (A3), because in this case all τ's are 1 and τ_3. In this case the selection rule is simply read; if $N_\nu + N_\iota =$ an odd number the process is forbidden.

B) *Evaluation method.*

In order to simplify H we substitude Fourier integral representations for the functions involved in (A1)

$$\bar{S}(X) = \frac{1}{(2\pi)^4} P \int dK e^{iKX} \frac{(i\gamma K - x)}{K^2 + x^2}$$

$$S^{(1)}(X) = \frac{1}{(2\pi)^3} \int dK e^{iKX} (i\gamma K - x) \delta(K^2 + x^2)$$

$$U_L^{(i)} = U_L e^{iP_L x^{(i)}}$$

where $iP_4 < 0$ or > 0 according as U_L contains the annihilation or creation of a meson. Thus we obtain

$$H_I = \frac{1}{(2\pi)^{4n+3}} \int_{-\infty}^{\infty} dx' \cdots dx^{(n)} e^{i(-K+K'+P_L)X'} e^{i(-K'+K''+P_M)X''} \cdots e^{i(-K^{(n+1)}+K^{(n)}+P_S)X^{(n)}}$$

$$\times U_L U_M \cdots U_T \int dK dK' \cdots dK^{(n)} S_p \{ (i\gamma K - x)\gamma_L (i\gamma K' - x) \gamma_M \cdots \gamma_S (i\gamma K^{(n)} - x) \gamma_T \}$$

$$\times \sum_{i=0}^{n} \left(\frac{1}{K^2 + x^2} \cdot \frac{1}{K'^2 + x^2} \cdots \cdot \frac{1}{K^{(i-1)2} + x^2} \cdot \frac{1}{K^{(i+1)2} + x^2} \cdots \frac{1}{K^{(n)2} + x^2} \right) \delta(K^{(i)2} + x^2). \tag{A7}$$

Various simplification can be introduced, firstly

$$\sum_{i=0}^{n} \left(\frac{1}{K^2 + x^2} \cdot \frac{1}{K'^2 + x^2} \cdots \frac{1}{K^{(n)2} + x^2} \right) \delta(K^{(i)2} + x^2)$$

$$= \int_{-1}^{1} da_1 \cdots \int_{-1}^{1} da_n (-1)^n \Gamma_n \delta^n (x^2 + \xi K^2 + \xi_1 K'^2 + \cdots + \xi_n K^{(n)2}).$$

where

$$\xi = \frac{1+a_1}{2}, \quad \xi_1 = \frac{1-a_1}{2} \frac{1+a_2}{2}, \quad \ldots, \quad \xi_{n-1} = \frac{1-a_1}{2} \frac{1-a_2}{2} \cdots \frac{1-a_{n-1}}{2} \frac{1+a_n}{2},$$

$$\xi_n = \frac{1-a_1}{2} \frac{1-a_2}{2} \cdots \frac{1-a_{n-1}}{2} \frac{1-a_n}{2},$$

and

$$\Gamma_n = (\xi \cdot \xi_1 \cdots \xi_n) \Big/ \left(\frac{1-a_1^2}{2} \cdot \frac{1-a_2^2}{2} \cdots \frac{1-a_n^2}{2} \right)$$

secondly

$$\frac{1}{(2\pi)^4}\int e^{i\alpha X'}dx'=\delta(\alpha).$$

Then H_I is reduced to

$$H_I=\frac{1}{(2\pi)^3}\int dKdK'\cdots dK^{(n)}\delta(K-K'-P_L)\delta(K'-K''-P_M)\cdots\delta(K^{(n-1)}-K^{(n)}-P_S)$$

$$\times U_LU_MU_N\cdots U_T\times S_p\{(i\gamma K-\varkappa)\gamma_L(i\gamma K'-\varkappa)\gamma_M\cdots\gamma_S(i\gamma K^{(n)}-\varkappa)\gamma_T\}$$

$$\times\int_{-1}^{1}da_1\cdots\int_{-1}^{1}da_n(-1)^n\Gamma_n\delta^n(\varkappa^2+\xi K^2+\xi_1K'^2+\cdots+\xi_nK^{(n)2}). \qquad \text{(A8)}$$

We then make the substitution

$$K=\tilde{K}+(\xi_1+\xi_2+\cdots+\xi_n)P_L+(\xi_2+\cdot+\xi_n)P_M+\cdots+\xi_nP_S=\tilde{K}+\alpha.$$

$$K'=\tilde{K}+(\xi_2+\xi_3+\cdots+\xi_n+\xi)P_M+(\xi_3+\cdots+\xi)P_N+\cdots+\xi P_S=\tilde{K}+\beta$$

....

$$K^{(n)}=\tilde{K}+(\xi+\xi_1+\cdots+\xi_{n-1})P_T+(\xi_1+\cdots+\xi_{n-1})P_L+\cdots+\xi_{n-1}P_R=\tilde{K}+\zeta.$$

Making use of the relations

$$K-K'=P_L,\ K'-K''=P_M,\cdots K^{(n-1)}-K^{(n)}=P_S$$

and the energy-momentum conservation law $P_L+\cdots+P_T=0$ (from this we obtain $K^{(n)}-K=P_T$) and $\xi+\xi_1+\cdots+\xi_{n-1}+\xi_n=1$. and $\xi\alpha+\xi_1\beta+\cdots+\xi_n\zeta=0$. We obtain

$$H_I=\frac{1}{(2\pi)^3}\int d\tilde{K}U_LU_M\cdots U_T\int_{-1}^{1}da_1\cdots\int_{-1}^{1}da_n(-1)^n\Gamma_n\delta^n(\varkappa^2+\tilde{K}^2+\xi\alpha^2+\xi_1\beta^2+\cdots+\xi_n\zeta^2)$$

$$\times S_p\{(i\gamma\cdot(\tilde{K}+\alpha)-\varkappa)\gamma_L(i\gamma\cdot(\tilde{K}+\beta)-\varkappa)\gamma_M\cdots\gamma_S(i\gamma\cdot(\tilde{K}+\zeta)-\varkappa)\gamma_T\}. \qquad \text{(A10)}$$

The integration over $\tilde{K}$ is now performed. We start from the next relation

$$\int d\tilde{K}e^{a\Delta}\delta(\Delta)=\int d\tilde{K}\delta(\Delta),\quad \text{where}\quad \Delta=\tilde{K}^2+A.$$

Differentiating the above formula with respect to A, we have

$$\int d\tilde{K}e^{a\Delta}(\delta'(\Delta)+a\delta(\Delta))=\int d\tilde{K}\delta'(\Delta)$$

or

$$\int dKe^{a\Delta}\delta'(\Delta)=\int d\tilde{K}\Big(\frac{d}{dA}-a\Big)\delta(\Delta).$$

Repeating this differentiation we obtain

$$\int d\tilde{K}e^{a\Delta}\delta^{(n)}(\Delta)=\int d\tilde{K}\Big(\frac{d}{dA}-a\Big)^n\delta(\Delta).$$

Expanding with respect to the power of a and comparing the coefficient of a^m, we obtain

$$\int d\tilde{K}\, \Delta^{n+a}\delta^n(\Delta)=0, \quad \text{for} \quad a\geq 0 \tag{A11}$$

$$\int d\tilde{K}\, \Delta^{n-a}\delta^n(\Delta)=(-1)^{n-a}\frac{n!}{a!}\int d\tilde{K}\,\delta^a(\Delta)d\tilde{K} \quad a\geq 0 \tag{A11'}$$

Then we use the next relation

$$\int d\tilde{K}\,\delta'(\Delta)=-\int d\boldsymbol{K}dk_0\frac{1}{2k_0}\frac{\partial}{\partial k_0}\delta(k_0^2-\boldsymbol{K}^2-A)$$

$$=-\frac{1}{2}\int\frac{d\boldsymbol{K}}{\sqrt{\boldsymbol{K}^2+A}^3}=-\pi\lim_{K\to\infty}\left\{\log\frac{(K+\sqrt{K^2+A})^2}{A}-2\right\}. \tag{A12}$$

Differentiating with respect to A, it becomes

$$\int d\tilde{K}\delta''(\Delta)=\pi/A \tag{A13}$$

Repeating this differentiation we have

$$\int d\tilde{K}\delta^n(\Delta)=(-1)^n\frac{(n-2)!}{A^{n-1}}\pi. \tag{A14}$$

Inserting (A14) into (A11′)

$$\int d\tilde{K}\Delta^{n-a}\delta^n(\Delta)=(-1)^n\frac{n!}{a(a-1)}\frac{\pi}{A^{a-1}}. \tag{A15}$$

Using (A11) (A12), (A13) (A14) and (A15) the matrixelement (A10) can be easily evaluated. The integration over $da_1\cdots da_n$ is not so simple, that we expand with respect to reciprocal power of nucleon mass $\varkappa$, assuming the meson mass are smaller than nucleon mass. Then the integration over $da_1\cdots$ becomes the combination of next simple integrations, which can be easily given as follows.

$$\int_{-1}^{1}da_1\int_{-1}^{1}da_2\cdots\int_{-1}^{1}da_n\,l'_n\xi_1^{\alpha}\xi_2^{\beta}\cdots\xi_n^{\eta}=\frac{\alpha!\;\beta!\cdots\eta!}{(\alpha+\beta+\cdots+\eta+n)!}. \tag{A16}$$

Thus we can evaluate the matrix element by the expansion of the reciprocal of nucleon mass far easier than ordinary perturbation method. There is no divergency and ambiguity in the matrix element of the decay of a heavier meson into four or more lighter mesons whereas they appear in lower order processes as mentioned above.

Note added in proof: The angular distribution of produced π meson by γ-ray is not $\sin^2\theta$ but nearly symmetrical, so the experimental data is in favour of pseudoscalar meson theory than scalar one. Comparison between various phenomenon of meson will be seen in Prog. Theor. Phys. (**5** (1950)), K. Aizu, Y. Fujimoto, H. Fukuda, S. Hayakawa, K. Takayanagi, G. Takeda and Y. Yamaguchi.

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On the Decay of a Heavy Meson into Lighter Mesons

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§ 1. Introduction.

Existence of a heavy meson (τ-meson) having the mass about 700~1000 times of an electron and the fact it disintegrates into π-mesons is almost confirmed by recent experiments. For example, G. D. Rochester and C. C. Butler[1] have reported the processes which are considered to be $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0}$ and $\tau^{0}\rightarrow\pi^{+}+\pi^{-}$ and C. F. Powell and others[2] have observed the decay process in which a τ-meson disintegrated into three π-mesons. Furthermore, L. Leprince-Ringuet[3] obtained the interesting photograph in which a τ^{-}-meson gave rise to star and a σ-meson, produced that star, led to the next star. Among them, Powell and his coworker's experiment seems to be most convincing and Leprince-Ringuet's photograph indicates us the fact that the coupling of τ-meson with nucleons is not so small because of the production of star. From the evidence mentioned above and the unstability of a τ-meson, it is reasonable to consider that the process occurs via nucleon field and a τ-meson is a Boson. If a τ-meson is a Fermion, Powell's process occurs as the following:

1) $\tau^{+}\rightarrow N+P+\tau^{0}\rightarrow\pi^{+}+P^{-}+P^{+}+\tau^{0}\rightarrow\pi^{+}+\pi^{+}+P^{-}+N+\tau^{0}$
$\rightarrow\pi^{+}+\pi^{+}+\pi^{-}+\tau^{0},$ (τ^{0} spin$\frac{1}{2}$)

2) $\tau^{+}\rightarrow P^{+}+P^{-}+\tau_{1}^{+}\rightarrow\pi^{+}+N+P^{-}+\tau_{1}^{+}\rightarrow\pi^{+}+\pi^{-}+\tau_{1}^{+}$ (τ_{1}^{+} spin$\frac{1}{2}$)

As for the case 1), the momentum law being satisfied by three particles (Powell's evidence), it is difficult to consider the other uncharged particle besides charged mesons and the mass of a τ-meson may be not so heavy that it disintegrates 4 particles. On the other hand, in the process 2), τ_{1}^{+} meson is not π, because it is a Fermion. Therefore in this case, we cannot help considering the new particle which has an almost same mass as that of a π-meson and spin $\frac{1}{2}$ and whose interaction with nucleon is fairly strong. Judging from this point of view, it seems to be reasonable to consider a τ-meson as a Boson.

If a τ-meson is a Boson, Powell's process occurs as the fourth order one in the perturbation method:

$$\tau^{+}\rightarrow P+N\rightarrow\pi^{+}+N+N\rightarrow\pi^{+}+\pi^{-}+P+N\rightarrow\pi^{+}+\pi^{-}+\pi^{+},$$

while the processes $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0}$ and $\tau^{\pm}\rightarrow\pi^{\pm}+\gamma$ occur generally more rapidly than the former, because of the third order process. Therefore, in order to explain Powell's experimental result, we ought to seek the forbidden case of $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0}$ and $\tau^{\pm}\rightarrow\pi^{\pm}+\gamma$. In the process $\tau^{\pm}\rightarrow\pi^{\pm}+\gamma$, the following cases are forbidden[4)]

1) when both τ and π meson have spin O.
2) when τ or π meson is the scalar meson with vector coupling.

In the process $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0}$,[5)] the cases of pseudoscalar τ meson and scalar and pseudoscalar π mesons ($ps\rightarrow s+s$, $ps\rightarrow ps+ps$) are forbidden for both symmetrical and neutral theory. From these results following choices for $\tau^{+}\rightarrow\pi^{+}+\pi^{-}+\pi^{+}$ are allowed, that is:

1) τ is pseudoscalar and π is pseudoscalar.
2) τ is pseudoscalar and π is scalar.

And from the selection rules for the fourth order transition process, we find that the all combinations of the former 1) are allowable and those of the latter 2) vanish. Besides the problem of searching for the decay schemes of τ meson consistent with experiments, there exist theoretically interesting problems concerning with the divergence, gauge invariance theorem for meson-nucleon- photon coupling. In this connection the examinations of the applicability of Pauli's regulator[6)] were made.

I The Decay of $\tau\rightarrow\pi+\gamma$.

§ 2. Method of Calculation and the General Hamiltonian Form.

The generalized Schrödinger equation for the system consisting of nucleons, photons, τ and π mesons becomes as follows:

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(X)}=\{H_1(X,\sigma)+H_2(X,\sigma)\}\Psi[\sigma], \tag{1}$$

$$H_1(X,\sigma)=H_G(\tau,n)+H_{Ge}(\tau,n,\gamma,\sigma),$$

$$H_2(X,\sigma)=H_e(n,\gamma)+H_e(\pi,\gamma)+H_e(\tau,\gamma)+H_g(\pi,n)+H_{ge}(\pi,n,\gamma,\sigma)$$

where $$H_e(n,\gamma)=-ic\bar{\psi}\tau_P\gamma_\mu\psi A_\mu,\quad \bar{\psi}=\psi^*\gamma_4, \tag{2}$$

$$H_G(\tau,n)=G_1U^*W+\frac{G_2}{\varkappa_\tau}\frac{\partial U^*}{\partial X_\alpha}P_\alpha+\frac{G_2^2}{\varkappa_\tau^2}2\pi P_\alpha^* P_\beta N_\alpha N_\beta+\text{c.c.} \tag{3}$$

$$H_e(\tau,\gamma)=\frac{1}{4\pi}\frac{ie}{\hbar c}A_\alpha\left(U^*\frac{\partial U}{\partial X_\alpha}-\frac{\partial U^*}{\partial X_\alpha}U\right)-\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2U^*U\{A_\alpha^2+(N_\alpha A_\alpha)^2\} \tag{4}$$

$$H_{Ge}(\tau,n,\gamma)=\frac{G_2}{\varkappa_\tau}\frac{ie}{\hbar c}A_\alpha(U^*P_\alpha-P_\alpha^*U)+\frac{G_2}{\varkappa_\tau}\frac{ie}{\hbar c}(A_\alpha N_\alpha P_\beta N_\beta U^*-A_\alpha N_\alpha P_\beta^* N_\beta U) \tag{5}$$

$$W=\bar{\psi}(i\gamma_5)\tau_{NP}\psi,\quad P_\alpha=i\bar{\psi}(\gamma_5)\gamma_\alpha\tau_{NP}\psi,\quad \varkappa_\tau=\frac{m_\tau c}{\hbar}$$

for the scalar (pseudoscalar) τ-meson.

For the vector (pseudovector) τ-meson, we get

$$H_G(\tau,n)=-G_1U_\alpha{}^*P_\alpha+\frac{G_2}{2\varkappa_\tau}U^*_{\alpha\beta}S_{\alpha\beta}+\frac{4\pi}{\varkappa_\tau^2}G_2^2\frac{1}{4}S_{\alpha\beta}S^*_{\alpha\beta}+\frac{4\pi}{\varkappa_\tau^2}\frac{1}{2}\{G_1^2P_\alpha{}^*P_\beta N_\beta N_\alpha$$
$$+G_2^2S^*_{\alpha\beta}S_{\alpha\gamma}N_\beta N_\gamma\}+\text{c.c.} \tag{6}$$

$$H_e(\tau,\gamma)=\frac{1}{8\pi}\Big[\frac{ie}{\hbar c}\{(A_\alpha U^*_\beta-A_\beta U^*_\alpha)U_{\alpha\beta}-U^*_{\alpha\beta}(A_\alpha U_\beta-A_\beta U_\alpha)\}$$
$$-\Big(\frac{ie}{\hbar c}\Big)^2\{(A_\alpha U_\beta{}^*-A_\beta U_\alpha{}^*)(A_\alpha U_\beta-A_\beta U_\alpha)$$
$$+2(A_\alpha U^*_\beta-A_\beta U^*_\alpha)(A_\alpha U_\mu-A_\mu U_\alpha)N_\beta N_\mu+\frac{2}{\varkappa_\tau^2}A_\alpha A_\beta U^*_{\alpha\mu}U_{\beta\nu}N_\mu N_\nu\}\Big] \tag{7}$$

$$H_{Ge}(\tau,n,\gamma)=-\frac{G_1}{\varkappa_\tau^2}\frac{ie}{\hbar c}\{A_\alpha U_{\alpha\beta}P_\gamma{}^*N_\beta N_\gamma-A_\alpha U^*_{\alpha\beta}P_\gamma N_\beta N_\gamma\}$$
$$+\frac{G_2}{2\varkappa_\tau}\frac{ie}{\hbar c}\{(A_\alpha U^*_\beta-A_\beta U^*_\alpha)S_{\alpha\beta}-(A_\alpha U_\beta-A_\beta U_\alpha)S^*_{\alpha\beta}+2(A_\alpha U^*_\beta-A_\beta U^*_\alpha)S_{\alpha\gamma}N_\gamma N_\beta$$
$$-2(A_\alpha U_\beta-A_\beta U_\alpha)S^*_{\alpha\gamma}N_\gamma N_\beta\} \tag{8}$$

$$S_{\alpha\beta}=-i\bar{\psi}(\gamma_5)\gamma_\alpha\gamma_\beta\tau_{NP}\psi,\qquad U_{\alpha\beta}=\frac{\partial U_\beta}{\partial X_\alpha}-\frac{\partial U_\alpha}{\partial X_\beta},$$

where N_α is the component of the normal to the surface σ and the other quantities express the usual meaning. For the π meson, we have only to replace $\varkappa_\tau$ by $\varkappa_\pi$ and G by g in the above expressions. The commutation relations between these field variables become as follows:

$$[U(X),\ U^*(X')]=4\pi i\hbar cD_\tau(X-X') \tag{9}$$

$$[U_\alpha(X),\ U^*_\beta(X')]=4\pi i\hbar c\Big(\delta_{\alpha\beta}-\frac{1}{\varkappa_\tau^2}\frac{\partial^2}{\partial X_\alpha\partial X_\beta}\Big)D_\tau(X-X') \tag{10}$$

$$\{\psi_\alpha(X),\ \bar{\psi}_\beta(X')\}=-iS_{\alpha\beta}(X-X')\tau_N-iS_{\alpha\beta}(X-X')\tau_P,$$
$$S_{\alpha\beta}(X-X')=\Big(\gamma_\mu\frac{\partial}{\partial X_\mu}-\varkappa\Big)\Delta(X-X'),\quad \varkappa=\frac{Mc}{\hbar}, \tag{11}$$

M=nucleon mass, other commutators=0.

As our process occurs through the coupling Gge, it is convenient to make the following canonical transformation

$$\Psi[\sigma]=U[\sigma]\Psi'[\sigma] \tag{12}$$

where $U[\sigma]$ satisfies the relation

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(X)}=H_2(X,\sigma)U[\sigma]. \tag{13}$$

By this transformation we get

$$i\hbar c\frac{\delta\Psi'[\sigma]}{\delta\sigma(X)}=U^{-1}[\sigma]H_1(X,\sigma)U[\sigma]\Psi'[\sigma]. \tag{14}$$

Then in the order Gge, our generalized Schrödinger equation reads

$$i\hbar c\frac{\delta\Psi'[\sigma]}{\delta\sigma(X)}=(P+Q+R)\Psi'[\sigma] \tag{15}$$

$$P=A+B,\quad A=-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega'[H_{Ge}(\tau,n,\gamma,\sigma),\,H_g'(\pi,n)] \tag{16}$$

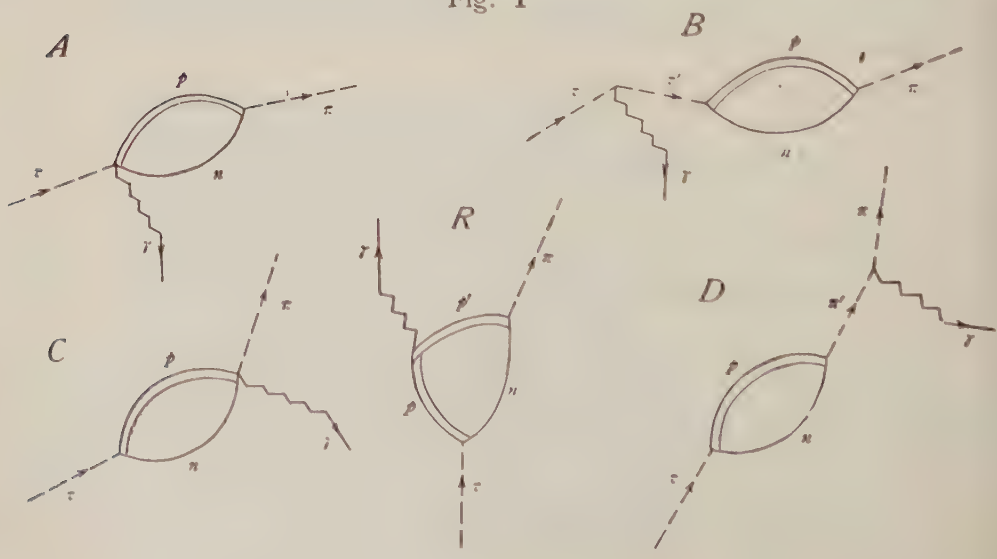
$$\begin{aligned}B=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\{[H_G(\tau,n),\,H_e'(\tau,\gamma)],\,H_g''(\pi,n)]\\+[[H_G(\tau,n),\,H_g'(\pi,n)],\,H_e''(\tau,\gamma)]\}\end{aligned} \tag{17}$$

$$Q=C+D,\quad C=-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega'[H_G(\tau,n),\,H_{ge}'(\pi,n,\gamma,\sigma)] \tag{18}$$

$$\begin{aligned}D=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\{[[H_G(\tau,n),\,H_e'(\pi,\gamma)],\,H_g''(\pi,n)]\\+[[H_G(\tau,n),\,H_g'(\pi,n)],\,H_e''(\pi,\gamma)]\}\end{aligned} \tag{19}$$

$$\begin{aligned}R=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\{[[H_G(\tau,n),\,H_g'(\pi,n)]H_e''(\pi,\gamma)]\\+[[H_G(\tau,n),\,H_e'(n,\gamma)],\,H_g''(\pi,n)]\}\end{aligned} \tag{20}$$

Fig. 1



According to the Feynman's diagram, these terms correspond to the processes shown in Fig. 1. P and Q are the processes where τ and π mesons interact with electromagnetic field respectively and moreover they contain the normal dependent terms. Owing to the Furry's theorem there are many cases where these terms vanish. Generally, it was shown by Z. Koba[8] that such normal dependent terms are just eliminated by the next higher effect of the Hamiltonian density. In this case the normal dependent terms of A and C are canceled out by one part of the term containing second derivatives in B and D respectively, i.e.,

$$\int_{-\infty}^{\sigma} \partial''_\alpha \partial''_\beta D(X-X'')F(X'')d\omega'' = N_\alpha N_\beta F(X) + \int_{-\infty}^{\sigma} d\omega'' D(X-X'')\partial''_\alpha \partial''_\beta F(X'').$$

Therefore the total expressions are independent of the assignment of the surfaces.

§ 3. Formal Proof of the Gauge Invariancy.

Though our proof may not be completely strict,[1] it is possible to show that whole expressions are invariant under the transformation

$$A_\alpha \to A_\alpha + \partial_\alpha \Lambda, \qquad (\Box \Lambda = 0). \tag{21}$$

1) The cases when P and Q vanish.

Using the relation $(\gamma\partial + \varkappa)S^{(1)}(X-X')=0$ and $(\gamma\partial+\varkappa)\bar{S}(X-X')=-\delta(X-X')$ and integrating by part, the term containing Λ in R becomes

$$R:\quad -\frac{i}{2}\frac{Gge}{(\hbar c)^2}\int_{-\infty}^{\infty} d\omega'(\Lambda-\Lambda')S_p\{S^{(1)}(X'-X)O_\pi \bar{S}(X-X')O_\tau + \bar{S}(X'-X)O_\pi S^{(1)}(X-X')O_\tau\}UV' \tag{22}$$

where $U=$ wave function of τ meson, $V=$ wave function of π meson, which is proved to be always zero in this case, since the integrand is similar to those of P and Q.

2) The cases when P and Q do not vanish.

In these cases, adding to the terms (22) the contributions from P and Q, the whole expressions can also be shown to be invariant under gauge transformation. For instance, in the case $G_2(ps)g_1(pv)e$ each terms containing Λ are given by

$$A:\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{i}{2}\int_{-\infty}^{\infty} d\omega' F(X-X')U\frac{\partial\Lambda}{\partial X_\alpha}V_\beta^{*\prime}$$

$$B:\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{-i}{2}\int_{-\infty}^{\infty} d\omega' F(X-X')\left(U\frac{\partial\Lambda}{\partial X_\alpha}+\frac{\partial U}{\partial X_\alpha}\Lambda\right)V_\beta^{*\prime}$$

$$D:\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{i}{2}\int_{-\infty}^{\infty} d\omega' F(X-X')\frac{\partial U}{\partial X_\alpha}\Lambda' V_\beta^{*\prime}$$

$$R: \quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{-i}{2}\int_{-\infty}^{\infty} d\omega' F(X-X')(A'-A)\frac{\partial U}{\partial X_\alpha}V_\beta^{*\prime}$$

where

$$F(X-X')=S_p\{S^{(1)}(X'-X)\gamma_5\gamma_\alpha\bar{S}(X-X')\gamma_5\gamma_\beta+\bar{S}(X'-X)\gamma_5\gamma_\alpha S^{(1)}(X-X')\gamma_5\gamma_\beta\}$$

which cancel each other. Summarizing the results of 1) and 2), the gauge invariance seems to be formally secured for every case.

§ 4. Formal Proof of Equivalence Theorem.

According to Case's general treatment of the equivalence theorem for photon-meson-nucleon coupling,[10] equivalence theorem is also expected because our case is of the order Gge.

To verify this, we put in R, $U\to\partial_\alpha U$, $O_\tau=\gamma_5\gamma_\alpha$ and $G=i\,\varkappa_\tau G_2$. U denotes the wave function of scalar (pseudoscalar) τ meson. For scalar (pseudoscalar) τ meson with vector (pseudovector) coupling, the usual procedure gives for R

$$\begin{aligned}R=-\frac{i}{2}\frac{Gge}{(\hbar c)^2}\int_{-\infty}^{\sigma} d\omega'\int_{-\infty}^{\sigma'} d\omega'' S_p\,\{&S^{(1)}(X-X'')O_\pi\bar{S}(X''-X')\gamma_\alpha\bar{S}(X'-X)O_\tau\\ &+\bar{S}(X-X'')O_\pi S^{(1)}(X''-X')\gamma_\alpha\bar{S}(X'-X)O_\tau\\ &+\bar{S}(X-X'')O_\pi\bar{S}(X''-X')\gamma_\alpha S^{(1)}(X'-X)O_\tau\}UA'_\alpha V'' \end{aligned}\qquad (23)$$

$R=0$ (*part equivalent to pseudoscalar coupling*)

$$\begin{aligned}&+\frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{1}{2}\Big[UV^*\int_{-\infty}^{\infty} d\omega' A'_\alpha S_p\{S^{(1)}(X-X')\gamma_\alpha\bar{S}(X'-X)(\gamma_5)M\\ &\qquad\qquad+\bar{S}(X-X')\gamma_\alpha S^{(1)}(X'-X)(\gamma_5)M\}\\ &+UA_\alpha\int_{-\infty}^{\infty} d\omega' V^{*\prime}S_p\{S^{(1)}(X-X')M\bar{S}(X'-X)\gamma_\alpha\gamma_5)+\bar{S}(X-X')MS^{(1)}(X'-X)\gamma_\alpha\gamma_5)\}\Big].\end{aligned}\qquad (24)$$

Using the regulator and removing the zeroth term in the expansion of the power $\Box/\varkappa^2$, the first term in the above bracket is zero by the relation $\Box A_\mu=0$. The second term, as is easily seen, cancels the first normal independent term in (16) exactly. For the remaining terms B and D identities are easily verified. Thus, formally identities hold if we admit the regulator and drop the undesirable terms. It should be, however, remembered that in these proof expressions which may contain divergence are removed by the law of conservation of energy and momentum or usual canonical transformation and so our procedure cannot be said completely rigorous.

§ 5. Calculations and the Use of Regulator Method.

Using the representations of the Δ and $\Delta^{(1)}$ function proposed by Schwinger[11] and resolving the Hamiltonian density for the order Gge into gauge and non-gauge

parts, it is shown that non-gauge part vanishes except the leading term in the expansion of $\Box/x^2$. For the typical cases we get the following expressions,

1) Case of $G_2(ps)g_1(pv)e$

$$(P+Q)=(P+Q)_1+(P+Q)_2$$

$$(P+Q)_1=\frac{G_2}{x_\tau}\frac{g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\frac{x^2}{2}\left[\left\{2\int_0^\infty\frac{\sin w}{w^2}dw+2\int_0^\infty\frac{\cos w}{w}dw+\frac{1}{3}\frac{x_\tau^2}{x^2}\right\}\frac{1}{x_\pi^2}UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha}\right.$$
$$\left.+\left\{\frac{2}{3}\frac{x_\pi^2}{x^2}\int_0^\infty\frac{\cos w}{w}dw+\frac{1}{3}(x_\pi^2+x_\tau^2)\right\}UA_\alpha V_\alpha^*\right]$$

$$(P+Q)_2=\frac{G_2}{x_\tau}\frac{g_1e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{1}{8}\frac{x_\tau^2}{x_\pi^2}\frac{x_\tau^4}{x^4}\int_{-1}^1\frac{y^2(y-1)}{1-\dfrac{1-y^2}{4}\dfrac{x_\tau^2}{x^2}}dy\,UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha},$$

$$R=R_1+R_2,$$

$$R_1=\frac{G_2}{x_\tau}\frac{g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\frac{1}{3}\left[x_\pi^2\int_0^\infty\frac{\cos w}{w}dwUA_\alpha V_\alpha^*-\frac{7}{6}x_\pi^2UA_\alpha V_\alpha^*-UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha}-x_\tau^2UA_\alpha V_\alpha^*\right]$$

$$R_2=\frac{G_2}{x_\tau}\frac{g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\frac{1}{2^4}\int_1^\infty\frac{du}{u^6}\int_{-1}^1dv\left\{1+\frac{(1-v)(1-u)}{2u^2}\frac{x_\pi^2}{x^2}-\frac{1-v^2}{4u^2}\frac{x_\tau^2}{x^2}\right\}^{-1}$$
$$\times\left[\frac{4}{3}(1-v)(-3+6u+5v-4uv-3u^2)\frac{x_\pi^2}{x^2}+(1-v^2)\left(4-4u-\frac{16}{3}v\right)\frac{x_\tau^2}{x^2}\right]UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha}$$

2) Case of $G_1(ps)g_1(pv)e$

$$(P'+Q')=(P'+Q')_1+(P'+Q')_2+(P'+Q')_3$$

$$(P'+Q')_1=\frac{G_1g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}x\int_0^\infty\frac{\cos w}{w}dwU\left(A_\alpha V_\alpha^*+\frac{1}{x_\pi^2}F_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha}\right),$$

$$(P'+Q')_2=\frac{G_1g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\frac{x}{6}\frac{x_\tau^2}{x^2}\frac{1}{x_\pi^2}F_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha},$$

$$(P'+Q')_3=\frac{G_1g_1e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{x}{16}\frac{1}{x_\pi^2}\frac{x_\tau^4}{x^4}\int_{-1}^1\frac{y^2(y^2-1)}{1-\dfrac{1-y^2}{4}\dfrac{x_\tau^2}{x^2}}dy\,UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha},$$

$$R'=R_1'+R_2'+R_3'$$

$$R_1'=\frac{G_1g_1e}{(\hbar c)^2}\frac{i}{(2\pi)^2}x\left[\int_0^\infty\frac{\cos w}{w}dw-\frac{1}{2}\right]UA_\alpha V_\alpha^*,$$

$$R_2'=\frac{G_1g_1e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{1}{6x}UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha},$$

$$R_3'=\frac{G_1g_1e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\cdot\frac{1}{8x}\int_0^\infty\frac{du}{u^6}\int_{-1}^1dv\frac{(1-v-u)\{-2(1-v)(1-u)x_\pi^2+(1-v^2)x_\tau^2\}}{1+\dfrac{(1-v)(1-u)}{2u^2}\dfrac{x_\pi^2}{x^2}-\dfrac{1-v^2}{4u^2}\dfrac{x_\tau^2}{x^2}}\frac{1}{x^2}$$
$$\times UF_{\alpha\beta}\frac{\partial V_\beta^*}{\partial X_\alpha}$$

Those terms which are divergent and non-gauge invariant can be removed by the following conditions imposed on regulator[12] which are written in the bracket.

$(P+Q)_1$; $\int \rho(x) x \log|x|\,dx=0$, $\int \rho(x)\log|x|\,dx=0$, $\int\rho(x)\,dx=0$.

$(P'+Q')_1$; $\int \sqrt{x}\,\rho(x)\,dx=0$,

R_1; $\int \rho(x)\,dx=0$, $\int\rho(x)\log|x|\,dx=0$,

R_1'; $\int\sqrt{x}\,\rho(x)\,dx=0$.

To secure the equivalence theorem we must remove the terms $(P'+Q')_2$ and R_2' which are gauge invariant and seem to be absolutely convergent. Therefore the equivalence theorem does not hold in this case.

§ 6. Selection Rules and the Life Times of these Processes.

After calculation, it turns out that the following combinations of τ and π mesons are forbidden.

1) The cases when both τ and π mesons have spin O.

2) The cases when τ or π meson is the scalar meson with vector coupling. In these cases, the use of regulator is made and the diverging and non-gauge invariant terms are removed.

The rule 1) is easily understood. Under the request of gauge invariancy the interaction Hamiltonians in these cases must be the linear combinations of $F_{\alpha\beta}\partial_\alpha U\partial_\beta\varphi^*$, $F_{\alpha\beta}\varphi^*\partial_\alpha\partial_\beta U$ and $F_{\alpha\beta}U\partial_\alpha\partial_\beta\varphi^*$ which are identically zero. Taking as the constants the following values and using the regulator

$$G_2^2/\hbar c\sim10^{-3},\quad g^2/\hbar c\sim 10^{-1},\quad \varkappa_\tau=900\,\varkappa_e,\quad \varkappa_\pi=300\,\varkappa_e,$$

the decay life times are as follows and shorter than 10^{-11} sec.

$G_1(v)g_1(v)e$	4.2×10^{-14} sec.	$G_2(v)g_2(ps)e$	2.6×10^{-12} sec.
$G_1(pv)g_2(pv)e$	2.3×10^{-13} sec.	$G_1(v)g_1(ps)e$	2.8×10^{-14} sec.
$G_1(ps)g_1(pv)e$	1.5×10^{-1}[illegible] sec.	$G_1(s)g_1(s)e$	4.0×10^{-15} sec.
$G_2(ps)g_2(pv)e$	1.3×10^{-13} sec.	$G_1(s)g_2(v)e$	3.1×10^{-13} sec.

II. The Decay of $\tau^\pm\to\pi^\pm+\pi^0$.[13]

§ 7. Method of Calculation.

For this case, the Hamiltonian density corresponding to (14) is the following:

$$\frac{Ggf}{2(\hbar c)^2}\iint_{-\infty}^{\infty} S_p\{S^{(1)}(X''-X)O_1\bar S(X-X')O_2\bar S(X'-X'')O_3$$
$$+\varepsilon S^{(1)}(X-X'')O_3\bar S(X''-X')O_2\bar S(X'-X)O_1$$

$$
\begin{aligned}
&+\bar{S}(X''-X)O_1S^{(1)}(X-X')O_2\bar{S}(X'-X'')O_3\\
&+\varepsilon\bar{S}(X-X'')O_3S^{(1)}(X''-X')O_2\bar{S}(X'-X)O_1\\
&+\bar{S}(X''-X)O_1\bar{S}(X-X')O_2S^{(1)}(X'-X'')O_3\\
&+\varepsilon\bar{S}(X-X'')O_3\bar{S}(X''-X')O_2S^{(1)}(X'-X)O_1\}\\
&\times U_1V_2'W_3''d\omega'd\omega''
\end{aligned}
\tag{25}
$$

where U_1, V_2 and W_3 are wave functions or their derivatives of Bosons, U_1 of initial and V_2 and W_3 of final and O_1, O_2 and O_3 are Dirac matrices. ε equals to 1 for neutral and -1 for symmetrical theory. Applying the Schwinger's expressions for the $\bar{S}$ and $S^{(1)}$ appeared in (25), (25) can be written in the following form

$$H=H_I+H_{II}+H_{III}+H_{IV}$$

$$
\begin{aligned}
H_I=&-\frac{\varkappa^3Ggf}{8(\hbar c)^2(2\pi)^{12}}S_p(O_1O_2O_3+\varepsilon O_3O_2O_1)\iiint_{-\infty}^{\infty}\Big[\int(dk^0)(dk^{0\prime})(dk^{0\prime\prime})\int_{-\infty}^{\infty}dadbdc\\
&\times Se(\xi,\eta)e(k_\nu^{0\prime\prime2})e(\varkappa^2)e(R)\Big]U_1(X)V_2(X+\xi)W_3(X-\eta)d\xi d\eta
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
H_{II}=&-\frac{i\varkappa^2Ggf}{8(\hbar c)^2(2\pi)^{12}}\iiint_{-\infty}^{\infty}\Big[\int(dk^0)(dk^{0\prime})(dk^{0\prime\prime})\int_{-\infty}^{\infty}dadbdcS\{S_p(O_1O_2\gamma_\alpha O_3-\varepsilon O_3\gamma_\alpha O_2O_1)L_\alpha\\
&+S_p(O_1\gamma_\alpha O_2O_3-\varepsilon O_3O_2\gamma_\alpha O_1)M_\alpha+S_p(\gamma_\alpha O_1O_2O_3-\varepsilon O_3O_2O_1\gamma_\alpha)N_\alpha\}\\
&\times e(\xi,\eta)e(k_\nu^{0\prime\prime2})e(\varkappa^2)e(R)\Big]U_1(X)V_2(X+\xi)W_3(X-\eta)d\xi d\eta
\end{aligned}
\tag{27}
$$

$$H_{III}=H_{III,A}+H_{III,B},\quad H_{IV}=H_{IV,A}+H_{IV,B},$$

$$
\begin{aligned}
H_{III,A}=&-\frac{\varkappa Ggf}{8(\hbar c)^2(2\pi)^{12}}\iiint_{-\infty}^{\infty}\Big[\int(dk^0)(dk^{0\prime})(dk^{0\prime\prime})\int_{-\infty}^{\infty}dadbdcS\{S_p(O_1\gamma_\alpha O_2\gamma_\beta O_3\\
&+\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1)M_\alpha L_\beta+S_p(\gamma_\beta O_1O_2\gamma_\alpha O_3+\varepsilon O_3\gamma_\alpha O_2O_1\gamma_\beta)N_\beta L_\alpha\\
&+S_p(\gamma_\beta O_1\gamma_\alpha O_2O_3+\varepsilon O_3O_2\gamma_\alpha O_1\gamma_\beta)M_\alpha N_\beta\}\\
&\times e(\xi,\eta)e(k_\nu^{0\prime\prime2})e(\varkappa^2)e(R)\Big]U_1(X)V_2(X+\xi)W_3(X-\eta)d\xi d\eta
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
H_{III,B}=&-\frac{\varkappa Ggf}{8(\hbar c)^2(2\pi)^{12}}S_p(O_1\gamma_\alpha O_2\gamma_\alpha O_3+\varepsilon O_3\gamma_\alpha O_2\gamma_\alpha O_1+\gamma_\alpha O_1\gamma_\alpha O_2O_3+\varepsilon O_3O_2\gamma_\alpha O_1\gamma_\alpha\\
&+\gamma_\alpha O_1O_2\gamma_\alpha O_3+\varepsilon O_3\gamma_\alpha O_2O_1\gamma_\alpha)\iiint_{-\infty}^{\infty}\Big[\int(dk^0)(dk^{0\prime})(dk^{0\prime\prime})\int_{-\infty}^{\infty}dadbdcSk_\beta^{0\prime\prime2}e(\xi,\eta)\\
&\times e(k_\nu^{0\prime\prime2})e(\varkappa^2)e(R)\Big]U_1(X)V_2(X+\xi)W_3(X-\eta)d\xi d\eta
\end{aligned}
\tag{29}
$$

where the siffix β of $k_\beta^{0\prime\prime2}$ does not mean summation

$$
\begin{aligned}
H_{IV,A}=&\frac{iGgf}{8(\hbar c)^2(2\pi)^{12}}S_p(\gamma_\beta O_1\gamma_\alpha O_2\gamma_\tau O_3-\varepsilon O_3\gamma_\tau O_2\gamma_\alpha O_1\gamma_\beta)\iiint_{-\infty}^{\infty}\Big[\int(dk^0)(dk^{0\prime})(dk^{0\prime\prime})\\
&\times\int_{-\infty}^{\infty}dadbdcSM_\alpha N_\beta L_\tau e(\xi,\eta)e(k_\nu^{0\prime\prime2})e(R)\Big]U_1(X)V_2(X+\xi)W_3(X-\eta)d\xi d\eta
\end{aligned}
\tag{30}
$$

$$H_{IV,B}=\frac{iG_{3}f}{8(hc)^2(2\pi)^{12}}\iiint_{-\infty}^{\infty}[\int(dk^{0})(dk^{0\prime})(dk^{0\prime\prime})\int_{-\infty}^{\infty}dadbdc\; S\{S_p(\gamma_\alpha O_1\gamma_\alpha O_2\gamma_\beta O_3-\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1\gamma_\alpha)L$$
$$+S_p(O_1\gamma_\alpha O_2\gamma_\alpha O_3\gamma_\beta-\varepsilon\gamma_\beta O_3\gamma_\alpha O_2\gamma_\alpha O_1)N_3+S_p(O_1\gamma_\beta O_2\gamma_\alpha O_3\gamma_\alpha-\varepsilon\gamma_\alpha O_3\gamma_\alpha O_2\gamma_\beta O_1)M_3\}$$
$$\times k_\gamma^{0\prime\prime 2}e(\xi,\eta)e(k_\nu^{0\prime\prime 2})e(x^2)e(R)]U_1(X)V_2(X+\xi)W(X+\eta)d\xi d\eta \tag{31}$$

where the suffix γ of $k_\gamma^{0\prime\prime 2}$ does not mean summation.

In these expressions the following abbreviations are used

$$S=\left(\frac{a}{|a|}+\frac{b}{|b|}\right)\left(\frac{c}{|c|}+\frac{a+b}{|a+b|}\right),\qquad e(\xi,\eta)=\exp i(k_\nu^0\xi_\nu+k_\nu^{\prime 0}\eta_\nu),$$

$$e(x^2)=\exp i(a+b+c)x^2,\qquad e(k_\nu^{0\prime\prime 2})=\exp i(a+b+c)k_\nu^{0\prime\prime 2},$$

$$e(R)=\exp(-i)\{ax_2^2+bx_3^2+(ak_\nu^0+bk_\nu^{0\prime})^2/(a+b+c)\},$$

$$L_\alpha=(ak_\alpha^0+bk_\alpha^{0\prime})/(a+b+c),\qquad M_\alpha=L_\alpha-k_\alpha^0,\quad N_\alpha=L_\alpha-k_\alpha^{0\prime}.$$

x is the reciprocal Compton wave length of a nucleon and x_2 and x_3 are that of V_2 and W_3 respectively. ξ and η stand for $X'-X$ and $X-X''$.

These terms are classified into two sets of different kinds, say, H_I, H_{III} and H_{II}, H_{IV} according to the number of matrices γ_μ contained in O_1, O_2 and O_3, for the spur of the product of odd number of γ_μ vanishes. In the case of a γ-decay of a charged meson, according to its non-symmetric character of the coupling with nucleons, only such terms remain that the order of O_1, O_2 and O_3 in the spur is $O_3O_2O_1$. $H_{III,B}$ and $H_{IV,B}$ are the terms containing logarithmic divergences. It is necessary to notice that H_I and H_{III} contains the odd powers of x and H_{II}, H_{IV} even powers of x. We use the symbol $[x^1]$ for the former and $[x^0]$ for the latter.

§ 8. Selection Rule.

In calculating the transition processes of the third order, making use of Hamiltonians from (26) to (31), it is found that there is a certain selection rule for the forbidden processes. For example, in the case of $\tau^\pm\to\pi^\pm+\pi^0$, assuming that $\pi^\pm$ and π^0 obey wave equations of the same type, we have 128 combinations in all. Among them, many terms vanish by taking into account the characters of spurs, forms of wave functions raised from the differentiation of $\bar{\Delta}$ and $\Delta^{(1)}$. Investigation of these vanishing terms shows that if the spur of H_I vanishes, then the spurs of H_{III} vanish and if the one of the spurs of H_{II} vanish, then the rest of H_{II} and the spurs of H_{IV} vanish. When spur of H_I vanishes identically owing to the existence of γ_5 (say, the case of the form $S_p(\gamma_5\gamma_\mu\gamma_\nu)$, it is necessary to examine one of the spurs of $H_{III,A}$. This fact is of very interesting and we have reached the following selection rules held for the forbidden processes. Namely,

$$[x^1]\quad S_p(O_1O_2O_3+\varepsilon O_3O_2O_1)U_1V_2W_3=0 \qquad \text{forbidden}$$
$$\neq 0 \qquad \text{allowed} \tag{32}$$

$$[x^0] \quad S_p(O_1O_2\gamma_\alpha O_3-\varepsilon O_3\gamma_\alpha O_2O_1)\,U_1\frac{\partial}{\partial X_\alpha}(V_2W_3)=0 \quad \text{forbidden} \tag{33}$$

$$\neq 0 \quad \text{allowed}$$

In the case where (32) vanish identically owing to γ_5, then

$$[x^1] \quad S_p(O_1\gamma_\alpha O_2\gamma_\beta O_3+\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1)\,U_1\frac{\partial^2(V_2W_3)}{\partial X_\alpha\partial X_\beta}=0 \quad \text{forbidden} \tag{34}$$

$$\neq 0 \quad \text{allowed}$$

This rules seems to hold usually for the third order processes of such type that the Boson interact symmetrically with nucleons. Together with the equivalence theorem stated in § 4, this rule indicates us the forbidden processes almost perfectly. Using the relation

$$S_p(\gamma_\alpha\gamma_\beta\cdots\cdots\gamma_\mu\gamma_\nu)=S_p(\gamma_\nu\gamma_\mu\cdots\cdots\gamma_\beta\gamma_\alpha),$$

almost the most part of the above selection rule is simplified as follows. Since the orders of matrices containing in the selection rules are opposite in the first and the second terms except the order of matrices involved in O_1, O_2 and O_3,

	$G_1g_1f_1$	$G_1g_1f_2$	$G_1g_2f_1$	$G_1g_2f_2$	$G_2g_1f_1$	$G_2g_1f_2$	$G_2g_2f_1$	$G_2g_2f_2$
$s\to s+s$	$ND(1)$	$SD(0)$	$SD(0)$	—*	$SD(0)$	—*	—*	$SD(0)$
$s\to ps+ps$	$ND(1)$	$ND(0)$	$ND(0)$	$ND(1)$	$SD(0)$	$SD(1)$	$SD(1)$	$SD(0)$
$ps\to s+s$	—	—	—	—	—	—	—	—
$ps\to ps+ps$	—	—	—	—	—	—	—	—
$s\to v+v$	$NF(1)$	$ND(0)$	$ND(0)$	$ND(1)$	$SD(0)$	$SD(1)$	$SD(1)$	$SD(0)$
$s\to pv+pv$	$ND(1)$	$SD(0)$	$SD(0)$	$ND(1)$	$SD(0)$	—*	—*	$SD(0)$
$ps\to v+v$	$NF(1)_0$	$ND(0)$	$ND(0)$	$ND(1)$	$NF(0)_0$	$NF(1)$	$NF(1)$	$ND(0)$
$ps\to pv+pv$	$NF(1)_0$	$SD(0)$	$SD(0)$	$ND(1)$	$NF(0)_0$	$SD(1)$	$SD(1)$	$ND(0)$
$v\to s+s$	$SD(0)$	—*	—*	$SD(0)$	—	—*	—*	$SD(1)$
$v\to ps+ps$	$SD(0)$	$SD(1)$	$SD(1)$	$SD(0)$	—	$SD(0)$	$SD(0)$	$SD(1)$
$pv\to s+s$	—	—	—	—	$SF(1)$	—*	—*	$SD(1)$
$pv\to ps+ps$	—	—	—	—	$SF(1)$	$SD(0)$	$SD(0)$	$SD(1)$
$v\to v+v$	$SD(0)$	$SD(1)$	$SD(1)$	$SD(0)$	$SD(1)$	$SD(0)$	$SD(0)$	$SF(1)$
$v\to pv+pv$	$SD(0)$	$NF(1)$	$NF(1)$	$SD(0)$	$SD(1)$	$ND(0)$	$ND(0)$	$SF(1)$
$pv\to v+v$	$NF(0)$	$NF(1)$	$NF(1)$	$ND(0)$	$SD(1)$	$SD(0)$	$SD(0)$	$SF(1)$
$pv\to pv+pv$	$NF(0)$	$NF(1)$	$SD(1)$	$ND(0)$	$SD(1)$	$ND(0)$	$ND(0)$	$SF(1)$

Table I.

when the order of γ-matrices included in O_1, O_2 and O_3 are interchanged in the second term of the rules, then the second term becomes equivalent to the first but for sign. As the O_i whose sign is altered by conversion of the order of its factors is $\gamma_\lambda\gamma_\mu$, $\gamma_\mu\gamma_\nu$ and $\gamma_\lambda\gamma_\mu\gamma_\nu$, therefore, the second term alters its sign when the number of $\gamma_\lambda\gamma_\mu$, $\gamma_\mu\gamma_\nu$, $\gamma_\lambda\gamma_\mu\gamma_\nu$, we call this number "$\boldsymbol{K}$", is odd and does not when $\boldsymbol{K}$ is even, if we make the order of γ-matrices of the selection rules more simply as follows:

$[x^1]$ *is forbidden when* $\mathbf{K}$ *is odd for* $\varepsilon=1$ *and* $\mathbf{K}$ *is even for* $\varepsilon=-1$.

$[x^0]$ *is forbidden when* $\mathbf{K}$ *is even for* $\varepsilon=1$ *and* $\mathbf{K}$ *is odd for* $\varepsilon=-1$.

However, as is easily seen, this rule is less satisfactory than the above and is, so to speak, the necessary condition and not the sufficient one.

The results of adopting these two relations are listed in Table I. In this table G and F stand for the coupling constants of a τ-meson with nucleons, g and f of a π meson. And, for example, in the case of a pseudoscalar meson, suffix 1 stands for its pseudoscalar coupling and suffix 2 its pseudovector one. S means that the term is allowed for symmetrical theory ($\tau=\tau_3$) and N for neutral one ($\tau=1$). It is easily seen that S and N are mutually exclusive. *Div.* and *Fin.* are abbreviations of "containing divergences" and "containing no divergences" respectively. Symbol * denotes that it vanishes by equivalence theorem.

Symbol o denotes that the equivalence theorem holds between those terms.

D ; *Div.* F ; *Fin.* (1) ; $[x^1]$ (0) ; $[x^0]$

§ 9. Decay Life Time.

Decay life time about several examples are calculated.

ex. 1 $G_1g_1f_1$ of $s\rightarrow ps+ps$

$$H=\frac{2\varkappa G_1g_1f_1}{(2\pi)^2(\hbar c)^2}UVW\int_0^\infty\frac{\cos w}{w}dw-\frac{\varkappa G_1g_1f_1}{(2\pi)^2(\hbar c)^2}UVW$$

$$+\frac{\varkappa G_1g_1f_1}{2(2\pi)^2(\hbar c)^2}\left(2\varkappa_\pi^2-\frac{\varkappa_\tau^2}{3}\right)\frac{1}{\varkappa^2}UVW+\cdots\cdots$$

The first term is dropped off by the condition $\int\rho(\varkappa)\log|\varkappa|\,d\varkappa=0$. The life time due to each term is the following:

$\tau_1^{-1}=T$ (from the second term) which is dropped off, if $\int\rho(\varkappa)\varkappa d\varkappa=0$

$\tau_2^{-1}=\frac{1}{4}\left(\frac{2\varkappa_\pi^2}{\varkappa^2}-\frac{\varkappa_\tau^2}{3\varkappa^2}\right)^2T,$ **is used.**

where

$$T=\frac{1}{8\pi^2}\frac{G_1^2g_1^2f_1^2}{(\hbar c)^3}\varkappa_\tau c\frac{\varkappa^2(\varkappa_\tau^2-4\varkappa_\pi^2)^2}{\varkappa_\tau^3}$$

ex. 2 $G_1 g_1 f_2$ of $s \to ps + ps$

$$H = -\frac{\varkappa G_1 g_1 f_2}{2(2\pi)^2(\hbar c)^2} \delta UVW\left(\frac{\varkappa_\pi^2}{\varkappa^2} - \frac{\varkappa_\tau^2}{\varkappa^2}\right)\int_0^\infty \frac{\cos w}{w} dw$$

$$+ \frac{\varkappa G_1 g_1 f_2}{2(2\pi)^2(\hbar c)^2} \delta\left(\frac{20}{3}\frac{\varkappa_\pi^2}{\varkappa^2} - \frac{23}{3}\frac{\varkappa_\tau^2}{\varkappa^2}\right) + \cdots\cdots$$

where

$$\delta = \frac{g_2}{g_1}\frac{2\varkappa}{\varkappa_\pi}.$$

The first diverging term is dropped off by the condition $\int \rho(\varkappa) \log|\varkappa| d\varkappa = 0$.

$$\tau_1^{-1} = \frac{1}{64}\delta^2\left(\frac{20}{3}\frac{\varkappa_\pi^2}{\varkappa^2} - \frac{23}{3}\frac{\varkappa_\tau^2}{\varkappa^2}\right)^2 T.$$

These numerical values are listed in Table II. As for the other interaction types, almost decay life time lies in general $10^{-14} \sim 10^{-18}$ sec.

ex. 1	(B)	3.5×10^{-13}	0.9×10^{-14}
	(A)	1.6×10^{-22}	0.3×10^{-18}
ex. 2	(B)		0.5×10^{-18}
	(A)		0.3×10^{-20}

Table II.

(A) is the one obtained by putting $g_2^2/\hbar c = 10^{-1}$ under the relation $g_1 = 2\varkappa/\varkappa_\pi g_2$.

(B) is the one without using the above relation.

The first column can be removed by the regulator $\int \rho(\varkappa)\varkappa d\varkappa = 0$. Therefore these processes must be prohibited.

III. The Decay of $\tau^\pm \to \pi^\pm + \pi^+ + \pi^-$.[14)]

§ 10. Method of Calculation.

As mentioned in II and III, the decay life time of $\tau \to \pi + \gamma$ and $\tau^\pm \to \pi^\pm + \pi^0$ lies in general $10^{-14} \sim 10^{-18}$ sec. Therefore, in order to explain Powell's experimental result, we ought to seek the forbidden case of the processes mentioned above. In the process $\tau \to \pi + \gamma$, the following cases are forbidden

1) when both τ and π mesons have spin 0.

2) when τ or π meson is the scalar meson with vector coupling.

In the process $\tau^\pm \to \pi^\pm + \pi^0$, the cases of pseudoscalar τ meson and scalar and pseudoscalar π mesons ($ps \to s + s$, $ps \to ps + ps$) are forbidden for both symmetrical and neutral theory. From these results following cases for $\tau \to 3\pi$ are allowed, i.e.

1) τ is pseudoscalar and π is pseudoscalar.

2) τ is pseudoscalar and π is scalar.

Next, we shall calculate the fourth order process. Hamiltonian density of this process has the form

$$\mathfrak{H}=g\bar{\psi}O_1\psi U_1+f\bar{\psi}O_2\psi V_2+h\bar{\psi}O_3\psi W_3+G\bar{\psi}O_4\psi Y_4$$
$$=H_B+H_A(=G\bar{\psi}O_4\psi Y_4) \tag{35}$$

where the same notations as part 1 are used. Y_4 is the wave function of τ meson and U_1, V_2 and W_3 are those of π mesons. The fundamental equation is

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(X)}=\mathfrak{H}(X)\Psi[\sigma]$$

and after the canonical transformation just like as part 1, we obtain

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(X)}=H(X)\Psi[\sigma]$$

where

$$H=\left(\frac{i}{\hbar c}\right)^3\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\int_{-\infty}^{\sigma''}d\omega'''\,[H_B''',[H_B''[H_B',H_A]]] \tag{36}$$

which is to be calculated. H_B''' etc. stand for $H_B(X''')$ etc., respectively. Performing the analogous calculation as in part I and II, we obtain the following expressions for H, that is,

$$H=\frac{Ggfh}{2(\hbar c)^3}\{H_1+H_2+H_3\} \tag{37}$$

where H_i $(i=1,2,3)$ has the general form

$$\begin{aligned}H_i=\int_{-\infty}^{\infty}Y_4U_1'V_2''W_3'''(d\omega)\sum S_p\{&S^{(1)}(X-X')K\bar{S}(X'-X'')L\bar{S}(X''-X''')M\bar{S}(X'''-X)N\\&-\frac{1}{4}\bar{S}(X-X')KS^{(1)}(X'-X'')LS^{(1)}(X''-X''')MS^{(1)}(X'''-X)N\\&+S^{(1)}(X'-X)N\bar{S}(X-X''')M\bar{S}(X'''-X'')L\bar{S}(X''-X')K\\&-\frac{1}{4}\bar{S}(X'-X)NS^{(1)}(X-X''')MS^{(1)}(X'''-X'')LS^{(1)}(X''-X')\}\end{aligned} \tag{38}$$

and H_1 is obtained by substitution

$$K\to O_1,\quad L\to O_2,\quad M\to O_3,\quad N\to O_4$$

H_2, by replacement $X'\to X''$, $X''\to X'$ in the spur and substitution

$$K\to O_2,\quad L\to O_1,\quad M\to O_4,\quad N\to O_3$$

and H_3 by $X''\to X'''$, $X'''\to X''$ in the spur and

$$K\to O_1,\quad L\to O_3,\quad M\to O_2,\quad N\to O_4$$

$(d\omega)$ means $d\omega'd\omega''d\omega'''$ and the wave function has the form $Y_4U_1'V_2''W_3'''$ in every H_i $(i=1,2,3)$. $\sum$ means summation over the four terms which is obtained by moving the position of $S^{(1)}$ in the first and the third terms or that of $\bar{S}$ in

the second and the fourth terms of the expression (38) to the right. Obviously, the second and the fourth terms vanish as they do not correspond to any real transition processes. Using the Schwinger's expression for $S^{(1)}$ and $\bar{S}$, we obtain, for example for H_1, the following expression

$$
\begin{aligned}
H_1=&\frac{i}{8(2\pi)^{16}}\int_{-\infty}^{\infty} Y_4(X)U_1(X-\zeta)V_2(X-\eta)W_3(X+\xi)(d\omega)\\
&\times\int(dk)(dk')(dk'')(dk''')\int_{-\infty}^{\infty} dadbdcde\\
&\times\left(\frac{a}{|a|}+\frac{b}{|b|}\right)\left(\frac{c}{|c|}+\frac{e}{|e|}\right)\left(\frac{a+b}{|a+b|}+\frac{c+e}{|c+e|}\right)\\
&\times S_p[(i\gamma_\alpha k_\alpha-\varkappa)O_1(i\gamma_\beta k_\beta'+\varkappa)O_2(i\gamma_\gamma k_\gamma''+\varkappa)O_3(i\gamma_\delta k_\delta'''-\varkappa)O_4\\
&+O_4(i\gamma_\delta k_\delta'''+\varkappa)O_3(i\gamma_\gamma k_\gamma''-\varkappa)O_2(i\gamma_\beta k_\beta'-\varkappa)O_1(i\gamma_\alpha k_\alpha+\varkappa)]\\
&\times\exp i\{k_\mu\zeta_\mu+k_\mu'(\zeta_\mu-\eta_\mu)+k_\mu''(\xi_\mu+\eta_\mu)+k_\mu'''\xi_\mu\}\\
&\times\exp i(a+b+c+e)\varkappa^2\exp i(ak_\mu^2+bk_\mu'^2+ck_\mu''^2+ek_\mu'''^2)
\end{aligned}
\tag{39}
$$

where $\xi=X'-X,\quad \eta'=X-X'',\quad \zeta=X-X'''$.

Now transforming k_μ's to k_μ^0's by the formulae

$$
\begin{aligned}
k_\mu&\rightarrow k_\mu^{0'''}+\{-ek_\mu^0+(c+e)k_\mu^{0'}+(b+c+e)k_\mu^{0''}\}/(a+b+c+e),\\
k_\mu'&\rightarrow k_\mu^{0'''}+\{ek_\mu^0-(c+e)k_\mu^{0'}+ak_\mu^{0''}\}/(a+b+c+e),\\
k_\mu''&\rightarrow k_\mu^{0'''}+\{ek_\mu^0+(a+b)k_\mu^{0'}+ak_\mu^{0''}\}/(a+b+c+e),\\
k_\mu'''&\rightarrow k_\mu^{0'''}+\{(a+b+c)k_\mu^0-(a+b)k_\mu^{0'}-ak_\mu^{0''}\}/(a+b+c+e),
\end{aligned}
$$

then (39)

$$
\begin{aligned}
H_1=&\frac{i}{8(2\pi)^{16}}\int_{-\infty}^{\infty} Y_4(X)U_1(X-\zeta)V_2(X-\eta)W_3(X+\xi)(d\omega)\int(dk^0)\int(dA)S\times S_p\cdots]\\
&\times\exp i(\xi_\mu k_\mu^0+\eta_\mu k_\mu^{0'}+\zeta_\mu k_\mu^{0''})\exp i(a+b+c+e)\varkappa^2\exp i(a+b+c+e)k_\mu^{0'''2}\\
&\times\exp(-i)\{(a+b+c)e\varkappa^2+(a+b)(c+e)\varkappa_2^2+a(b+c+e)\varkappa_1^2+2(a+b)ck_\mu^0k_\mu^{0'}\\
&+2ack_\mu^0k_\mu^{0''}-2a(c+e)k_\mu^{0'}k_\mu^{0''}\}/(a+b+c+e).
\end{aligned}
\tag{40}
$$

$S_p[\cdots\cdots]$ in (40) is the same as in (30) except the fact that the k_μ's are replaced by k_μ^0's, (dA) and (dk^0) stand for $dadbdcde$ and $(dk^0)(dk^{0'})(dk^{0''})(dk^{0'''})$ respectively and

$$S=\left(\frac{a}{|a|}+\frac{b}{|b|}\right)\left(\frac{c}{|c|}+\frac{e}{|e|}\right)\left(\frac{a+b}{|a+b|}+\frac{c+e}{|c+e|}\right).$$

k_μ^0's satisfy the conditions:

$$k_\mu^{0\,2}+\varkappa_3^2=0,\quad k_\mu^{0'2}+\varkappa_2^2=0,\quad k_\mu^{0''2}+\varkappa_1^2=0 \tag{41}$$

that is, k^0_μ's have the meaning of the momenta and energies of π-mesons. Then, we transform the variables a, b, c, e to u, v, z, w by

$$a=\frac{1}{2x^2}wu(u-1)(1-v),\qquad b=\frac{1}{2x^2}wu(u-1)(1+v),$$

$$c=\frac{1}{2x^2}wu(1+z),\qquad b=\frac{1}{2x^2}wu(1-z).$$

After these transformations, we obtain

$$(dA)=\frac{1}{4x^8}u^4(u-1)w^2\,|w|\,dudvdzdw,\quad S=8\frac{w}{|w|}$$

and the domain of these variables are determined as follows:

$$-1<v<1,\quad -1<z<1,\quad 1<u<\infty,\quad -\infty<w<\infty.$$

$S_p[\cdots\cdots]$ is divided into five parts; $S_p[\cdots\cdots]=\sum_{i=0}^{4}\Gamma_i$, and Γ_i is the term proportional to x^i. Finally, we obtain for H_1 the next expression

$$H_1=\frac{i}{4(2\pi)^{16}x^8}\int_{-8}^{\infty}Y_4(X)U_1(X-\zeta)V_2(X-\eta)W_3'(X'-\xi)(d\omega)$$
$$\times\int(dk^0)\int_{-\infty}^{\infty}w^3dw\int_{-1}^{1}dv\int_{-1}^{1}dz\int_{1}^{\infty}du$$
$$\times u^4(u-1)\sum_{i=0}^{4}\Gamma_i\exp\left(i\frac{wu^2}{x^2}k^{0''2}_\mu\right)\exp iwu^2(1-f) \tag{42}$$

where

$$f=\frac{(2u+z-1)(1-z)}{4u^2}\frac{x_3^2}{x^2}+\frac{u-1}{u^2}\frac{x_2^2}{x^2}+\frac{(u-1)(1-v)(uv+u-v+1)}{4u^2}\frac{x_1^2}{x^2}$$
$$+\frac{(u-1)(1-z)}{u^2}\frac{k^0_\mu k^{0'}_\mu}{x^2}+\frac{(u-1)(1-v)(1-z)}{2u^2}\frac{k^0_\mu k^{0''}_\mu}{x^2}-\frac{(u-1)(1-v)}{u^2}\frac{k^{0'}_\mu k^{0''}_\mu}{x^2}.$$

Employing the abbreviations

$$A_\mu=-\frac{1-z}{2}k^0_\mu+k^{0'}_\mu+\frac{1}{2}(uv+u-v+1)k^{0''}_\mu,$$

$$B_\mu=\frac{1-z}{2}k^0_\mu-k^{0'}_\mu+\frac{1}{2}(u-1)(1-v)k^{0''}_\mu,$$

$$C_\mu=\frac{1-z}{2}k^0_\mu+(u-1)k^{0'}_\mu+\frac{1}{2}(u-1)(1-v)k^{0''}_\mu,$$

$$D_\mu=\frac{1}{2}(2u+z-1)k^0_\mu-(u-1)k^{0'}_\mu-\frac{1}{2}(u-1)(1-v)k^{0''}_\mu.$$

Γ_i can be written, in the required approximation, in the following forms,

$$
\begin{aligned}
\Gamma_0 = {} & S_p(\alpha O_1\beta O_2\gamma O_3\delta O_4 + O_4\delta O_3\gamma O_2\beta O_1\alpha)\, k_\alpha^{0'''} k_\beta^{0'''} k_\gamma^{0'''} k_\delta^{0'''} \\
& -\frac{1}{u^2}[S_p(\alpha O_1\beta O_2\gamma O_3\delta O_4 + O_4\delta O_3\gamma O_2\beta O_1\alpha)\, C_\gamma D_\delta \\
& \quad - S_p(\gamma O_1\beta O_2\alpha O_3\delta O_4 + O_4\delta O_3\alpha O_2\beta O_1\gamma)\, A_\gamma D_\delta \\
& \quad + S_p(\delta O_1\beta O_2\gamma O_3\delta O_4 + O_4\alpha O_3\gamma O_2\beta O_1\delta)\, A_\delta C_\gamma \\
& \quad + S_p(\alpha O_1\gamma O_2\beta O_3\delta O_4 + O_4\delta O_3\beta O_2\gamma O_1\alpha)\, B_\gamma D_\delta \\
& \quad - S_p(\alpha O_1\delta O_2\gamma O_3\beta O_4 + O_4\beta O_3\gamma O_2\delta O_1\alpha)\, B_\delta C_\gamma \\
& \quad + S_p(\gamma O_1\delta O_2\alpha O_3\beta O_4 + O_4\beta O_3\alpha O_2\delta O_1\gamma)\, A_\gamma B_\delta]\, k_\alpha^{0'''} k_\beta^{0'''}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_1 = {} & -i\varkappa[S_p(\alpha O_1\beta O_2 O_3\delta O_4 - O_4\delta O_3 O_2\beta O_1\alpha)\,(k_\alpha^{0'''} k_\delta^{0'''} B_\beta - k_\alpha^{0'''} k_\beta^{0'''} \Gamma_\delta - k_\beta^{0'''} k_\delta^{0'''} A_\alpha) \\
& - S_p(\alpha O_1\beta O_2\gamma O_3 O_4 - O_4 O_3\gamma O_2\beta O_1\alpha)\,(k_\beta^{0'''} k_\gamma^{0'''} A_\alpha - k_\alpha^{0'''} k_\beta^{0'''} C_\gamma - k_\alpha^{0'''} k_\gamma^{0'''} B_\beta) \\
& - S_p(O_1\beta O_2\gamma O_3\delta O_4 - O_4\delta O_3\gamma O_2\beta O_1)\,(k_\beta^{0'''} k_\delta^{0'''} C_\gamma - k_\beta^{0'''} k_\gamma^{0'''} D_\delta - k_\gamma^{0'''} k_\delta^{0'''} B_\beta) \\
& + S_p(\alpha O_1 O_2\gamma O_3\delta O_4 - O_4\delta O_3\gamma O_2 O_1\alpha)\,(k_\alpha^{0'''} k_\delta^{0'''} C_\gamma - k_\alpha^{0'''} k_\gamma^{0'''} D_\delta - k_\gamma^{0'''} k_\delta^{0'''} A_\alpha)]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2 = {} & \varkappa^2[S_p(\alpha O_1\beta O_2 O_3 O_4 + O_4 O_3 O_2\beta O_1\alpha)\,(-k_\alpha^{0'''} k_\beta^{0'''} + \frac{1}{u^2} A_\alpha B_\beta) \\
& + S_p(O_1\beta O_2 O_3\delta O_4 + O_4\delta O_3 O_2\beta O_1)\,(-k_\beta^{0'''} k_\delta^{0'''} + \frac{1}{u^2} B_\beta D_\delta) \\
& - S_p(O_1\beta O_2\gamma O_3 O_4 + O_4 O_3\gamma O_2\beta O_1)\,(k_\beta^{0'''} k_\gamma^{0'''} + \frac{1}{u^2} B_\beta C_\gamma) \\
& - S_p(\alpha O_1 O_2 O_3\delta O_4 + O_4\delta O_3 O_2 O_1\alpha)\,(k_\alpha^{0'''} k_\delta^{0'''} + \frac{1}{u^2} A_\alpha D_\delta) \\
& + S_p(\alpha O_1 O_2\gamma O_3 O_4 + O_4 O_3\gamma O_2 O_1\alpha)\,(-k_\alpha^{0'''} k_\gamma^{0'''} + \frac{1}{u^2} A_\alpha C_\gamma) \\
& + S_p(O_1 O_2\gamma O_3\delta O_4 + O_4\delta O_3\gamma O_2 O_1)\,(-k_\gamma^{0'''} k_\delta^{0'''} + \frac{1}{u^2} C_\gamma D_\delta)]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_3 = {} & i\varkappa^2[S_p(O_1\alpha O_2 O_3 O_4 - O_4 O_3 O_2\alpha O_1)\, B_\alpha - S_p(\alpha O_1 O_2 O_3 O_4 - O_4 O_3 O_2 O_1\alpha)\, A_\alpha \\
& - S_p(O_1 O_2 O_3\alpha O_4 - O_4\alpha O_3 O_2 O_1)\, D_\alpha + S_p(O_1 O_2\alpha O_3 O_4 - O_4 O_3\alpha O_2 O_1)\, C_\alpha]
\end{aligned}
$$

$$
\Gamma_4 = \varkappa^4 S_p(O_1 O_2 O_3 O_4 + O_4 O_3 O_2 O_1)
$$

where α, β, γ, δ appeared in spur stand for γ_α, γ_β, γ_γ, γ_δ respectively. To calculate these, we will use the following formulae:

$$
\int \exp\,(iak_\mu^{0'''2})\,(dk^{0'''}) = \frac{i\pi^2}{a|a|}, \quad \int k_\mu^{0'''} k_\nu^{0'''} \exp\,(iak_\mu^{0'''2})\,(dk^{0'''}) = -\frac{\pi^2}{2a^2|a|}\delta_{\mu\nu}.
$$

$$
\int k_\alpha^{0'''} k_\beta^{0'''} k_\gamma^{0'''} k_\delta^{0'''} \exp\,(iak_\mu^{0'''2})\,(dk^{0'''})
$$

$$
= -\frac{6i\pi^2}{16a^3|a|}(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta^*_{\alpha\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta^*_{\alpha\beta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta^*_{\alpha\beta} + \delta_{\alpha\beta}\delta_{\beta\gamma}\delta_{\gamma\alpha}) \qquad (43)
$$

and

$$\int_{-\infty}^{\infty}\frac{w}{|w|}\exp(iwa)\,dw=\frac{2i}{a},\quad \int^{\infty}\frac{w^2}{|w|}\exp(iwa)\,dw=-\frac{2}{a^2},$$

where $\delta^{\times}_{\alpha\beta}=1$ when $\alpha\neq\beta$ and $=0$ when $\alpha=\beta$.

These calculations are very complicated, so that we perform the computation after the expansion with regard to the order x_π^2/x^2 is done.

§ 11. Selection Rule and Possible Type of the Processes.

Investigating the spur more precisely analogous to the case $\tau^{\pm}\to\pi^{\pm}+\pi^0$ and separating Γ_i's into two groups $[x^0]$, $[x^1]$ (notations are the same as in part II), then the selection rules become quite analogous to the case treated in part II; that is,

$$[x^0]\quad S_p(O_1O_2O_3O_4+O_4O_3O_2O_1)Y_4U_1V_2W_3=0 \qquad \text{forbidden}$$
$$\neq 0 \qquad \text{allowed} \tag{44}$$
$$[x^1]\quad S_p(uO_1O_2O_3O_4+O_4O_3O_2O_1u)Y_4\frac{\partial}{\partial X_\alpha}(U_1V_2W_3)=0 \qquad \text{forbidden}$$
$$\neq 0 \qquad \text{allowed}$$

Of course, the same cautions as in part II are required when the form $S_p(\gamma_\beta\gamma_\alpha\gamma_5)$ appears. If we wish to express this rule in another language, then it becomes as follows;

$$\begin{array}{lll} [x^0] & n=odd & \textit{forbidden} \\ & n=even & \textit{allowed} \\ [x^1] & n=even & \textit{forbidden} \\ & n=odd & \textit{allowed} \end{array} \tag{45}$$

n is the number of $\gamma_5\gamma_\alpha$, $\gamma_\alpha\gamma_5$ and $\gamma_\beta\gamma_\alpha\gamma_5$ appeared in O_1, O_2, O_3 and O_4. As was mentioned in § 10, the possible types of τ and π mesons are 1) both τ and π are pseudoscalar and 2) τ is pseudoscalar and π is scalar. Applying the rule to these two cases, we find that the all combinations of the former are allowable and those of the latter vanish. Therefore we have to calculate only the latter case where τ and π are of pseudoscalar.

§ 12. Life Time.

As $[x^0]$ contains logarithmically divergent term, we must drop off it and conditionally convergent term by using regulator $\int\rho(x)\log|x|dx=0$, $\int\rho(x)dx=0$ and $\int x\rho(x)dx=0$. Therefore, as for $[x^0]$, we have to calculate the terms proportional to the order of x_π^2/x^2. On the other hand, it is unnecessary to use regulators for $[x^1]$.

1) $G_1 g_1^3$ $[x^0]$

For the sake of comparison, life times are calculated for scalar and pseudoscalar. The third formula of (43) becomes

$$\int k_\alpha^{0'''} k_\beta^{0'''} k_\gamma^{0'''} k_\delta^{0'''} \exp\,(iak_\mu^{0'''2})\,(dk^{0'''}) = -\frac{i\pi^2}{4a^3|a|}(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}).$$

After applying the regulators, H_1 becomes as following

$$H_1=\frac{YUVW}{(2\pi)^2}\Bigg[\left\{\frac{14}{5}\frac{x_\pi^2}{x^2}-\frac{1}{15}\frac{x_\tau^2}{x^2}+\frac{12}{15}(2M+N-2L)\frac{x_\tau^2}{x^2}\right\}$$

$$-\int_0^\infty \frac{u-1}{u^6}du\int_{-1}^1 dv\int_{-1}^1 dz\frac{1}{x^2}(2C_\alpha D_\alpha-2A_\alpha D_\alpha-A_\alpha C_\alpha-B_\alpha D_\alpha-2B_\alpha C_\alpha+2A_\alpha B_\alpha)$$

$$+\begin{cases}-\dfrac{48}{10}\left\{\dfrac{11}{12}\dfrac{x_\tau^2}{x^2}-\dfrac{1}{12}\dfrac{x_\pi^2}{x^2}+\dfrac{1}{6}(2M+N-2L)\dfrac{x_\tau^2}{x^2}\right\} & \text{(scalar)}\\[2ex] +\dfrac{16}{10}\left\{\dfrac{11}{12}\dfrac{x_\pi^2}{x^2}-\dfrac{1}{12}\dfrac{x_\tau^2}{x^2}+\dfrac{1}{6}(2M+N-2L)\dfrac{x_\tau^2}{x^2}\right\} & \text{(pseudoscalar)}\end{cases}$$

$$+\begin{cases}\displaystyle\int_1^\infty \frac{u-1}{u^6}du\int_{-1}^1 dv\int_{-1}^1 dz\frac{1}{x^2}(A_\alpha B_\alpha+B_\alpha D_\alpha-B_\alpha C_\alpha-A_\alpha D_\alpha+A_\alpha C_\alpha+C_\alpha D_\alpha) & \text{(scalar)}\\[2ex] \displaystyle\int_1^\infty \frac{u-1}{u^6}du\int_{-1}^1 dv\int_{-1}^1 dz\frac{1}{x^2}(-A_\alpha B_\alpha+B_\alpha C_\alpha+B_\alpha D_\alpha+A_\alpha D_\alpha+A_\alpha C_\alpha-C_\alpha D_\alpha) & \text{(pseudoscalar)}\end{cases}$$

$$+\frac{8}{10}\left\{\frac{11}{12}\frac{x_\pi^2}{x^2}-\frac{1}{12}\frac{x_\tau^2}{x^2}+\frac{1}{6}(2M+N-2L)\frac{x_\tau^2}{x^2}\right\}\Bigg]$$

where the first and the second term come from Γ_0, the third and the fourth term from Γ_2 and the fifth term from Γ_4. And we use the following replacement together with (41) assuming that initial τ-meson is at rest and each π mesons have equal masses, that is,*

$$k_\mu^{0'} k_\mu^{0''}=\frac{1}{2}(x_\pi^2-x_\tau^2)+Lx_\tau^2, \qquad \sqrt{k_3^2+x_\pi^2}=Lx_\tau,$$

$$k_\mu^{0} k_\mu^{0'}=\frac{1}{2}(x_\pi^2-x_\tau^2)+Mx_\tau^2, \qquad \sqrt{k_1^2+x_\pi^2}=Mx_\tau,$$

---

* Law of conservation of energy and momentum can be written in the form

$$\vec{k}_1+\vec{k}_2+\vec{k}_3=0, \quad L+M+N=1.$$

In the integration of M and N, that lower limits can be determined from the values obtained by putting $\vec{k}_1=0$. That is, the lower limit of M becomes a. On the other hand, the upper limit can be decided as the value when $k_2=k_3$ and $\vec{k}_1$ is antiparallel to $\vec{k}_2$, $2k_2=k_1$. When M is given, the upper and lower limits of N are given by $(1-M+A)/2$, $(1-M-A)/2$ respectively, where $A=\sqrt{\dfrac{(M^2-a^2)(1-3a^2-2M)}{1+a^2-2M}}$. Therefore, the upper limit of M becomes $(1-3a^2)/2$.

$$k_\mu^0 k_\mu^{0\prime}=\frac{1}{2}(x_\tau^2-x_\mu^2)+Nx_\tau^2,\qquad \sqrt{k_\mu^2+x_\pi^2}-Nx_\tau.$$

After performing elementary calculation, we get

$$H_1=\frac{YUVW}{15(2\pi)^2}\left\{-18\frac{x_\pi^2}{x^2}+3\frac{x_\tau^2}{x^2}-(12M+6N)\frac{x_\tau^2}{x^2}\right\}\qquad \text{scalar}$$

$$H_2=\frac{YUVW}{15(2\pi)^2}\left\{79\frac{x_\pi^2}{x^2}-41\frac{x_\tau^2}{x^2}+(44M+58N)\frac{x_\tau^2}{x^2}\right\}\qquad \text{pseudoscalar}$$

As it is proved that H_1, H_2 and H_3 have the same contributions, total Hamiltonian density becomes

$$H=\frac{Gg^3}{40\pi^2(\hbar c)^3}YUVW\left(\frac{x_\tau}{x}\right)^2\{-18a^2+3-(12M+6N)\}\qquad \text{scalar}$$

$$H=\frac{Gg^3}{40\pi^2(\hbar c)^3}YUVW\left(\frac{x_\tau}{x}\right)^2\{79a^2-41+(44M+58N)\}\qquad \text{pseudoscalar}$$

where $\qquad a=x_\pi/x_\tau$.

Therefore, the required life time are obtained as follows;

$$\tau_0^{-1}=\frac{(Gg^3)^2x_\tau c}{400\pi^3(\hbar c)^4}\left(\frac{x_\tau}{x}\right)^4\int_a^{(1-3a^2)/2}dM\int_{(1-M-A)/2}^{(1-M+A)/2}dN\,\{-18a^2+3-(12M+6N)\}^2\qquad \text{scalar}$$

$$\tau_0^{-1}=\frac{(Gg^3)^2x_\tau c}{400\pi^3(\hbar c)^4}\left(\frac{x_\tau}{x}\right)^4\int_a^{(1-3a^2)/2}dM\int_{(1-M-A)/2}^{(1-M+A)/2}dN\,\{79a^2-41+(44M+58N)\}^2\qquad \text{pseudoscalar}$$

Substituting the values

$$g^2/\hbar c=10^{-1},\quad G^2/\hbar c=10^{-3},\quad M_\tau=1000\,m_e,\quad M_\pi=300\,m_e$$

we obtain

$$\tau_0=6.9\times10^{-12}\ \text{sec.}\qquad \text{for scalar}$$

$$\tau_0=1.5\times10^{-12}\ \text{sec.}\qquad \text{for pseudoscalar}$$

2) $G_2g_1^3$ $[x^1]$

It is unnecessary to use the regulator in this case. H becomes

$$H=\frac{G_2g_1^3}{8\pi^2(\hbar c)^3}\frac{x_\tau}{x}(2M+3N-1)YUVW$$

and

$$\tau_0^{-1}=\frac{9(G_2g_1^3)^2x_\tau c}{144\pi^3(\hbar c)^4}\left(\frac{x_\tau}{x}\right)^2\int_a^{(1-3a^2)/2}dM\int_{(1-M-A)/2}^{(1-M+A)/2}dN\,(2M+3N-1)^2$$

$$\tau_0=1.8\times10^{-12}\ \text{sec.}$$

3) $G_1g_2^3$ $[x^1]$

$$H=\frac{G_1 g_2^3}{8\pi^2(\hbar c)^3}\frac{\varkappa_\tau}{\varkappa}\left(\frac{\varkappa_\tau}{\varkappa_\pi}\right)^3 YUVW\Big[M^2+10MN+4N^2-(8+19a^2)M$$

$$-\left(\frac{19}{2}+a^2\right)N+5+\frac{5}{3}a^2-\frac{a^4}{4}\Big],$$

$$\tau_0^{-1}=\frac{9(G_1 g_2^3)^2\varkappa_\tau c}{144\pi^3(\hbar c)^4}\left(\frac{\varkappa_\tau}{\varkappa}\right)^2\left(\frac{\varkappa_\tau}{\varkappa_\pi}\right)^6\int_a^{(1-3a^2)/2}dM\int_{(1-M-A)/2}^{(1-M+A)/2}dN\Big[M^2+10MN+4N^2-(8+19a^2)M$$

$$-\left(\frac{19}{2}+a^2\right)N+5+\frac{5}{3}a^2-\frac{a^4}{4}\Big]^2.$$

$$\tau_0=1\times 10^{-14}\ \text{sec.}$$

In this case, life time does not become longer than in scalar case, because the term proportional to $[\varkappa^1]$ vanishes.

4) $G_2 g_2^3\ [\varkappa^0]$

According to the difficulties of calculation, we have used the relation

$$H_{pv}=\frac{2\varkappa}{\varkappa_\tau}\frac{G_2}{G_1}H_{pv}+\quad\text{third order term.}$$

Calculating the second term of right hand side, we find that it is about 160 times smaller than the first term and the life time of $G_2 g_2^3$ are able to decide by only the first term. If we put $G_2=G_1$, then, as $(2\varkappa/\varkappa_\tau)^2=13$, we get

$$\tau_0=7.7\times 10^{-16}\ \text{sec.}$$

If we adopt the values of constants

$$M_\tau=900\ m_e,\qquad M_\pi=286\ m_e,$$

life times become longer about the order of one or two than mentioned above. In this cases 3) and 4) are too short to be considered as reasonable values and can not explain Powell's experiment. Therefore, types of interaction seem to be pseudoscalar.

§ 13. Conclusions.

We see in I to III there are few cases in which a τ meson is possible to disintegrate into three π mesons with the suitable life time by dropping off diverging and non-gauge covariant terms by regulator method, although regulator method has the temporary meaning to remedy the inevitable ambiguity of the quantum field theory. Heavier meson observed by Rochester and Butler is not considered to be τ meson, because if this decay modes $\tau^\pm\to\pi^\pm+\pi^0$ or $\tau^\pm\to\pi^\pm+\gamma$ were correct, then we can hardly obtain any appropriate set with the life time of two particle decay as long as three particle decay, unless any new condition to lower the decay probability be added. And although for the processes considered the perturbation method was used, the other phenomena, i.e. two γ decay of a neutretto,[15] the angular distribution of produced π meson by γ-rays in Berkeley experiments[16] are also suitably explained by the results obtained

from perturbational treatment. Recent experiment observed by Panofsky[17] and Steinberger[18] shows the two γ decay of a neutretto, which proves to be the pseudoscalar property of neutral π meson.[19] Furthermore, the angular distribution of produced charged π meson by γ rays[16] is explained only by the model of pseudoscalar π meson. Therefore it may be concluded that these phenomena give the direct evidences for pseudoscalar meson with the process of $\tau \rightarrow 3\pi$, besides the indirect evidence, namely, the nuclear force favourable to pseudoscalar type.

Acknowledgment.

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Theory of Color Centers in Ionic Crystals. II.

—*The Temperature dependence of F-bands in Alkali Halides.*—

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The temperature dependence of *F*-absorption bands has been theoretically studied. It has been concluded that the main cause of the shift of the absorption peak is the thermal expansion of the lattice. As to the half-breadth of the band we calculated its temperature dependence by using the Einstein's model of lattice vibration. Quantitative results are in good agreement with Mollwo's experiments in the wide range of temperature for various kinds of alkali-halides.

§ 1. Introduction:

In Part I of this paper, the authors discussed the electronic energy levels of a trapped electron at the *F*-center in alkalihalide crystals. The gap between the ground level and the excited one corresponds to the peak wave length of the so called *F*-absorption band. Here we conside: its shape and temperature dependence theoretically.

In 1933, Mollwo[1] reported the observed data of absorption bands for some alkali-halide crystals over a wide range of temperature. His result for the case of KBr crystal is reproduced in Fig. 1. We can see in this diagram the following three distinct characteristics. The first characteristic is that the peak of the band shifts to the longer side of the wave length with the increasing temperature. The second is that the absoption curve has a nearly symmetrical bell-shape, and the third characteristic is that the band has a considerably wide half-width which broadens as the temperature increases. This wide width of the observed spectrum suggests the existence of a strong coupling between the trapped electron and the lattice vibration, so that we may find the key to solve our problem by analyzing the mechanism of this interaction. Considering the energy surface in configuration space, Mott-Gurney[2] and Kubo[3] already treated this problem from the phenomenological standpoint. Recently,

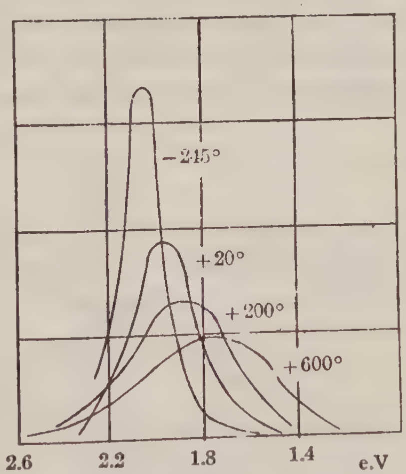

Fig. 1. Absorption band of KBr (Observed by Mollwo). The diagram is the reduced one from the direct observation in order to normalize the area of the curve.

Muto[4] reported some discussions from the atomistic view, in which he assumed the Mott-Froehlich's type of interaction between an electron and a lattice. The aim of this report is to obtain some quantitative results about the temperature dependence of absorption bands from the same standpoint as chosen in Part I where we replaced the crystal by a model of an equivalent large molecule.

To mention our basic idea briefly: In the first place we shall employ an adiabatic approximation because an electronic mass is very small compared with an ionic mass. Then the energy states of the electron are determined by solving the Schroedinger equation of the electronic system involving coordinates of ions as parameters. The deformation of the lattice causes the variation of the energy gap. For instance, we may take displacements of ions as parameters which describe this defo mation. In spite of the transition of a trapped electron we can conclude in the first approximation from Frank-Condon's principle that the lattice does not make a change of its vibrational state during the absorption process of light. According to this principle, absorption takes place where the wave length of light coincides with the energy gap determined as a function of displacement of ions. Thus, to a definite wave length of absorption there corresponds a certain displacement of ions. We assume that the intensity of absorption is proportional to the probability that the displacement takes such a definite value. This probability must be a statistical average taken over several states of vibrations. Concerning the shift of the peak with the temperature, we may regard that it is due to the thermal expansion of the lattice, In view of our standpoint, both the expansion and vibration of the lattice induce a small variation of electronic Hamiltonian, so that we may treat these effects as perturbation.

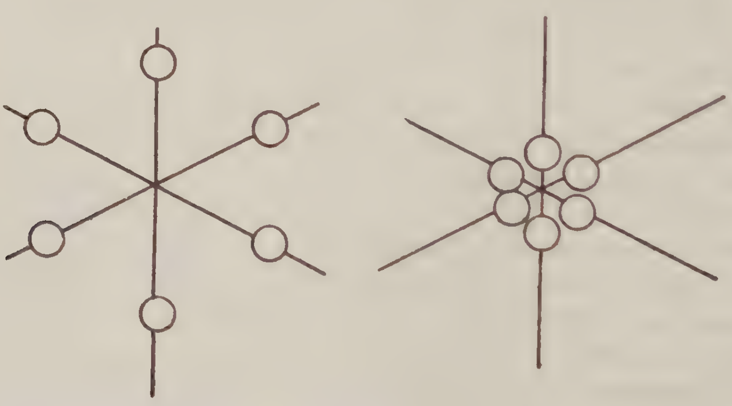

Fig. 2. The schematic representation of the lattice vibration.

Now we propose a simple model of the lattice vibration. Following the idea of the equivalent large molecule, we first consider the vibration of only the six nearest neighbours, i.e. six positive ions. Owing to a vacancy at the origin, the potential energy of the neighbouring ions may be smallest along the radial

direction. Therefore we can assume that the ionic oscillation is allowed only in this direction. The frequency of the vibration is mainly determined by the curvature of this potential curve. If we consider a weak coupling between the two opposite ions, we find two modes of vibrations, one mode is symmetrical and the other is antisymmetrical. As to the polarization effect, the latter has no contribution so that we may take into account only the former. By assuming this simple model, we can replace the lattice vibration by a linear harmonic oscillation. Fig. 2 shows these consideration schematically. Since our model is very simple it seems difficult to expect highly accurate results from our theory. In the following sections we intend to formulate this basic idea and we shall make a rough estimation of this problem.

§ 2. Perturbation energy.

Using the model of the rigid lattice, we have obtained in Part I a value of the energy gap which is in good agreement with the observation data. This suggests that the potential $V(\vec{r})$ may be described fairly well by the superposition of potentials due to point charges. In this picture, the energy of the trapped electron is determined by one parameter d, i.e. the inter-ionic distance. There are two possible causes which induce the variation of d. The one is thermal expansion and the other is the difference among composing ions. In this simplification there is no essential difference between these two cases. Taking the variation δd of d, the corresponding variation of energy gap δF is given by (2.1).

$$(\delta F/F)_a = \sigma(\delta d/d), \tag{2.1}$$

$$\sigma = (d/F)\frac{\partial}{\partial d}F(d). \tag{2.2}$$

If any precise result about electronic energy levels has been obtained by the method used in Part I, the accurate value of σ may be calculated by using (2.2). As we have already mentioned in Part I, Mollwo[5] has proposed the experimental formula

$$F = Kd^{-2}$$

which gives fairly good agreement with the data of various alkali halide crystals. So any calculated value of σ in good approximation is expected to lie near -2. The negative sign means that the absorption band displaces to the longer side of the wave length, with the increasing temperature.

Next we consider the perturbation induced by the displacement of the six nearest neighbours. Let the potential function $V(\vec{r})$ of our electron system be expressed by the following expanded form:

$$V(\vec{r}) = \sum_{all}{}' V_i(\vec{r}-\vec{R}_i) = \sum_{all}{}' V_i(\vec{r}-\vec{R}_{i0}) + \sum_{i=1}^{6}\delta\vec{R}_i \cdot grad_{Ri} V_i(\vec{r}-\vec{R}_{i0}) \tag{2.3}$$

where $\vec{R}_{i0}$ denotes the normal position of i-th ion, and $\delta\vec{R}_i$ denotes its displacement. In virtue of our model stated in the introduction of this paper, the perturbed potential δV is represented as follows:

$$\delta V=6\Big[\frac{\partial}{\partial R}V_1(\vec{r}-\vec{R}_{10})\Big]\delta R \tag{2.4}$$

where δR means the radial displacement of a nearest neighbour. Using an ordinary perturbation method, the following expression for the variation of the gap is obtained:

$$(\delta F/F)_{\delta R}=\sigma'(\delta R/d) \tag{2.5}$$

where

$$\sigma'=(d/F)\int(\delta V/\delta R)(\overset{(2)}{\Psi}_0\overset{(2)}{\Psi}_0-\overset{(1)}{\Psi}_0\overset{(1)}{\Psi}_0)d\tau. \tag{2.6}$$

Like the case of σ, we may calculate the accurate value of σ' using the results of Part I. But here we content ourselves with a rough estimate of its value and avoid to proceed in a direct way.

In the case of lattice expansion, we can take the perturbed Hamiltonian δH_d as follows:

$$\delta H_d=\sum_{eff}{}'\delta\vec{d}_i\cdot z_i grad_d 1/(\vec{r}-\vec{R}_{i0}) \tag{2.7}$$

where $\delta\vec{d}_i$ means displacement vector of i-th ion which corresponds to the lattice expansion, and that z_i is an effective charge explained in § 2 of Part I. Concerning the six nearest neighbours, we can further put the next relation:

$$\delta\vec{d}_i\cdot grad_d[1/(\vec{r}-\vec{R}_{i0})]=\delta d\Big[\frac{\partial}{\partial R_1}V_1(\vec{r}-\vec{R}_{10})\Big],\qquad i\leq 6 \tag{2.8}$$

Then if we replace the field of the effective large molecule in (2.7) with that of the six neighbours, we may approximate (2.7) as its rough estimate:

$$\delta H_d=\sum{}'z_i\delta\vec{d}_i\cdot grad_{\vec{d}_i}V_i(\vec{r}-\vec{R}_{i0})=6z_1'(\delta V/6\delta R)\delta d \tag{2.9}$$

by introducing the new factor z_1'. From (2.1) (2.2) (2.6) (2.9) we can get the relation between σ and σ'.

$$\begin{aligned}\sigma&=(d/F)\int(\delta H_d/\delta d)(\overset{(2)}{\Psi}_0\overset{(2)}{\Psi}_0-\overset{(1)}{\Psi}_0\overset{(1)}{\Psi}_0)d\tau\\&=(d/F)z_1'\int(\delta V/\delta R)(\overset{(2)}{\Psi}_0\overset{(2)}{\Psi}_0-\overset{(1)}{\Psi}_0\overset{(1)}{\Psi}_0)d\tau=z'\sigma'.\end{aligned} \tag{2.10}$$

It is difficult to determine the value of z_1', so we rather employ the proper value of σ' comparing with the experimental data as shown in § 4.

§ 3. Lattice vibration and the shape of the absorption band.

In the introduction of the paper, we explained our simple model of lattice vibration. The Schroedinger equation of the equivalent oscillator is given by

$$(d^2\Phi/d\eta^2)+(\varepsilon-a^2\eta^2)\Phi=0 \tag{3.1}$$

with

$$\eta=(\delta R/d),\quad a=4\pi^2M^+\nu d^2h^{-1} \tag{3.2}$$

where η means the displacement of a neighbour ion and a is a parameter describing the property of the crystal, in which M^+ is the mass of the positive ion, and ν is the vibration frequency and d is the inter-ionic distance. The solution of (3.1) is obtained as follows:

$$\begin{aligned}&\varepsilon=(2l+1)a \qquad l=0,\ 1,\ 2\cdots\cdots\\&\Phi_l(\xi)=(a/\pi)^{1/4}(1/2^l l\,!)e^{-\xi^2/2}H_l(\xi)\\&H_l(\xi)=(-)^l e^{\xi^2}\frac{d^l}{d\xi^l}(e^{\xi^2}),\qquad \xi=\sqrt{a}\,\eta.\end{aligned} \tag{3.3}$$

According to our fundamental assumption, variable ξ in (3.3) is proportional to the gap variation $\delta F/F$, so that it corresponds to the abscissa of the absorption curve.

Thus, $I(\xi,\theta)$, the intensity of absorption, is represented by the formula

$$I(\xi,\theta)\approx N\cdot f\cdot\sum_{l=0}^{\infty}\Phi_l^2(\xi)e^{-(l+\frac{1}{2})\theta}/\sum_{l=0}^{\infty}e^{-(l+\frac{1}{2})\theta} \tag{3.4}$$

where

$$\theta=h\nu/kT.$$

N is the concentration of F-centers and hence it is independent of the temperature if the vacancies are frozen in crystal. f means the transition probability of the trapped electron, so that in the first approximation we may calculate its value from the results of an unperturbed system:

$$f=f_0=\int z\overset{(2)}{\Psi}_0\overset{(1)}{\Psi}_0 d\tau. \tag{3.5}$$

The main part of (3.4) consists of the probability $\Phi_l^2(\xi)$, and the temperature dependence appears in the form of a statistical average over various excited states of lattice vibrations. A plot of (3.4) has a symmetrical bell-shape, and its peak corresponds to the value $\xi=0$. In absolute 0°K, θ tends to ∞ and the form of (3.4) is the same one as the Gaussian distribution function, so its half-width is $2\sqrt{ln2}$. Half-width $2<\xi>$ at any temperature is determined by solving the next equation (3.6) numerically,

$$\sum_{l=0}^{\infty}\Phi_l(0)e^{-(l+\frac{1}{2})\theta}=2\sum_{l=0}^{\infty}\Phi_l^2(<\xi>)e^{-(l+\frac{1}{2})\theta} \tag{3.6}$$

The curve in Fig. 4 shows the relation of $<\xi>$ to the temperature parameter θ^{-1}. We can also introduce the relation between the peak intensity and θ as follows,

$$I(0,\theta)/I(0,\infty)=(1-e^{-\theta})\sum_{l=0}^{\infty}\Phi_l^2(0)e^{-l\theta} \tag{3.7}$$

where $I(0,\infty)$ means the peak intensity at absolute $0°K$. Fig. 5 shows the plot of (3.7).

In these simple treatments, material characteristics appear in all quantities only through the parameters α, θ, etc. This is the reason why we reach to equations of reduced form like (3.4) (3.6) (3.7).

§ 4. Comparison with experiments.

First we estimate the shift of the peak wave length caused by the thermal expansion of the lattice. Using the data of thermal expansion coefficients for $\delta d/d$ from the expression of form

$$(\delta d/d_0)=at+bt^2+ct^3 \tag{4.1}$$

where t is temperature in °C, and for d_0 we take the value at 0°C. Coefficients a, b, c, are shown in Table 1, with a' for NaCl at a very low temperature

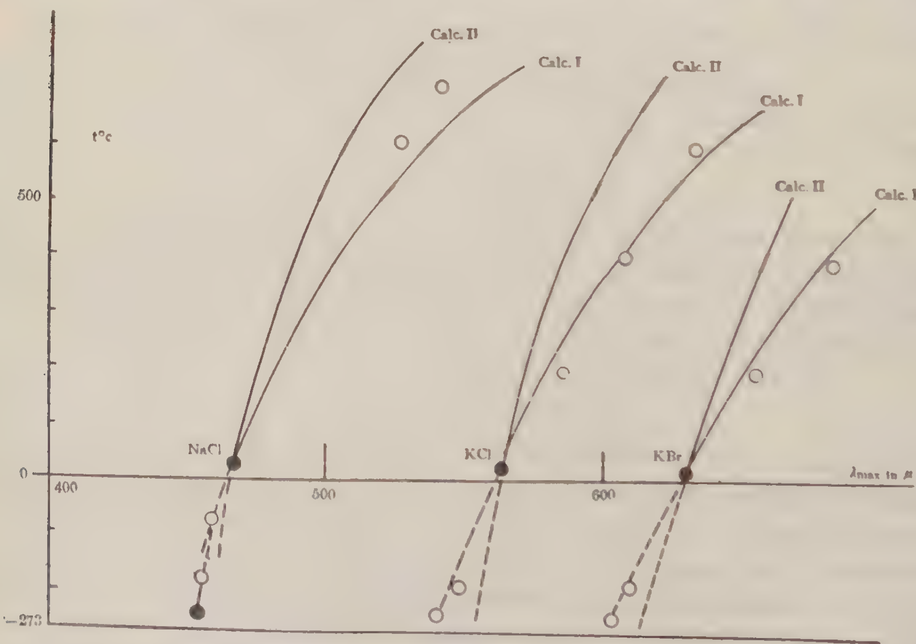

Fig. 3. The temperature effect of the peak wave length.
Calc. I. Calculated values for $g=2$
Calc. II. Calculated values for $g=1$
The dotted lines are results of extrapolations.

Because of a vacancy at the origin, we may expect a somewhat larger displacement than expected from (4.1). Specially if we introduce a factor g determined by the formula $\delta R = g\delta d$ in the case of thermal expansion, we get from (2.1) and (2.5) the following formula relating to the shift of the peak.

$$(\delta F/F) = \sigma[1 + (\sigma'/\sigma)(g-1)](\delta d/d). \tag{4.2}$$

Table 2 and Fig. 3 show the results of calculations compared with the observed data. Calc. I corresponds to the case of $g=2$ and Calc. II is for $g=1$, where we assume $\sigma=-2$, $\sigma'=-2$. We adjust the calculated value to the observed one at 20°C and estimate the relative shift from this temperature. For NaCl we adjust also the values at -253°C for the sake of using the coefficient a' at a low temperature. From these results it seems reasonable to regard the cause of the shift as mainly due to the thermal expansion of the lattice.

For the estimation of $<\xi>$ we need the value of frequency ν. Concerning the value ν_n for perfect crystals, Mott-Gurney[6] proposed the following formula.

	NaCl	KCl	KBr
°C	20~747	20~712	20~681
$a \times 10^6$	40.31	31.92	37.99
$b \times 10^9$	3.71	33.21	12.63
$c \times 10^{12}$	63.46	19.17	52.56
°C	-253~-193		
$a' \times 10^6$	10.8		

Table 1. The linear expansion coefficients of perfect crystals.

Temperature effect of peak wave length.
λ_{max} in μ

		t	−253	−186	− 79	− 20	+600	+700
NaCl	Obs.	λ_{max}	454	455	459	466	525	540
	Calc. I	λ_{max}	(454)	455	457	(466)	538	561
	Calc. II	λ_{max}	(454)	455	461	(466)	502	514
		t	−245	−186	+ 20	+200	+400	+600
KZl	Obs.	λ_{max}	540	548	563	584	605	630
	Calc. I	λ_{max}	542	546	(563)	579	605	641
	Calc. II	λ_{max}	553	554	(563)	571	584	602
		t	−245	−186	+ 20	+200	+400	
KBr	Obs.	λ_{max}	602	609	630	652	680	
	Calc. I	λ_{max}	602	608	(630)	649	680	
	Calc. II	λ_{max}	616	619	(630)	640	655	

Table 2. The temperature effect of the peak wave length. t is the temperature in °C. Calc. I. Calculated values for $g=2$. Calc. II. Calculated values for $g=1$. The values in parentheses are adjusted ones

	NaCl	KCl	KBa	KJ	RbCl
$M^{+}\times 6.02\times 10^{23}$g.	23.0	39.1	39.1	39.1	85.5
d $\times 10^{8}$ cm.	2.8	3.14	3.29	3.53	3.27
ρ $\times 10^{8}$ cm.	0.326	0.316	0.326	0.351	0.356
ν_n $\times 10^{-12}$ sec^{-1}.	7.29	5.19	4.89	4.39	3.14
$\sqrt{a_n}$	36.2	44.4	44.2	45.9	52.3

Table 3. Vibration frequency and others.

$$\nu_n = (1/2\pi)(2a_M/3)^{1/2}(d/\rho - 2)^{1/2}(e^2/M^+d^3)^{1/2} \tag{4.3}$$

where a_M is Madelung constant, and ρ is the parameter concerning the repulsive potential of ions determined from the data of compressibility. Replacing ν by ν_n in (3.2), we can estimate the value of a_n. These values are tabulated in Table 3. Owing to a vacancy at the origin, the frequency ν may be somewhat smaller than ν_n and we introduce a factor s for describing this effect:

$$\nu = s\nu_n < \nu_n, \quad a = sa_n < a_n. \tag{4.4}$$

In Table 4 Mollwo's observed data of half-width $<\delta F>/F_{\max}$ are shown over a wide range of temperature. By reducing the temperature to θ and $<\delta F>/F_{\max}$ to $<\xi>$ the observed values are expected to be on the one curve determined from (3.6).

Reducing formulas are as follows:

$$\theta = h\nu/kT = s(h\nu_n/kT) \tag{4.5}$$

$$2<\xi> = \sqrt{sa_n}\,|\sigma'|^{-1}. \quad (<\delta F>/F_{\max}) \tag{4.6}$$

The chosen value for s is tabulated in Table 5. The results are in good agreement with experiments over

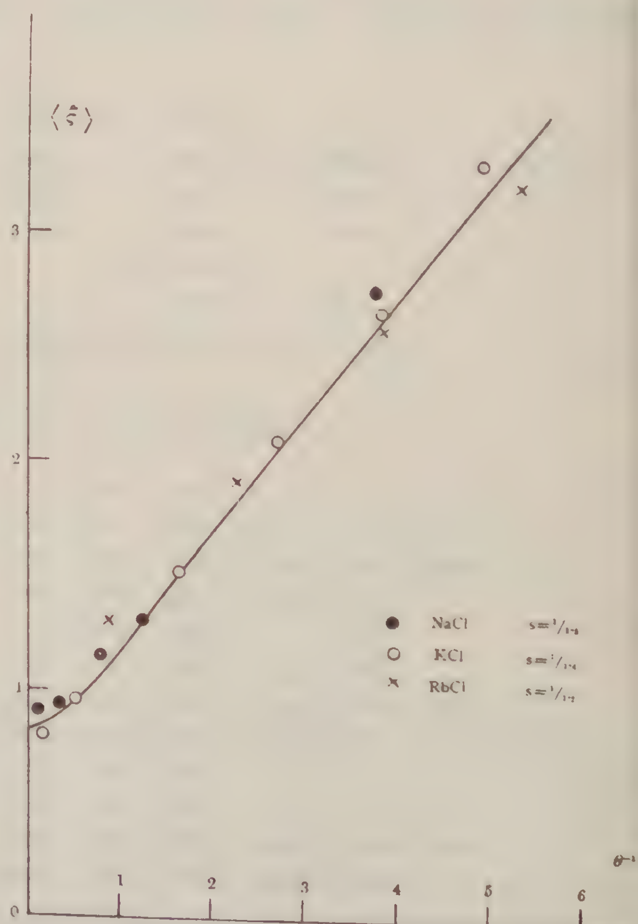

Fig. 4. The relation between the half-width and temperature. The curve is the theoretical result obtained from (3.6).

a wide range of temperature for various kinds of crystals as shown in Table 4, and Fig. 4, where we used the same numerical value for σ' as the case of thermal expansion (4.2).

	t	-253	-186	-79	$+20$	$+600$	
	$<\delta F>/F_{max}$	0.124	0.126	0.151	0.178	0.371	
NaCl	θ^{-1}	0.086	0.374	0.831	1.256	3.740	
	Obs. $<\xi>$	0.92	0.93	1.12	1.32	2.74	
	Calc. $<\xi>$	0.85	0.91	1.11	1.32	(2.59)	
	t	-245	-186	$+20$	$+200$	$+400$	$+600$
	$<\delta F>/F_{max}$	0.084	0.104	0.163	0.221	0.281	0.355
KCl	θ^{-1}	0.157	0.489	1.646	2.657	3.780	4.903
	Obs. $<\xi>$	0.79	0.98	1.53	2.08	2.64	3.33
	Calc. $<\xi>$	0.86	0.96	1.53	2.04	(2.60)	(3.16)
	t	-160	$+20$	$+200$	$+400$		
	$<\delta F>/F_{max}$	0.109	0.162	0.215	0.268		
RbCl	θ^{-1}	0.900	2.333	3.767	5.347		
	Obs. $<\xi>$	1.30	1.93	2.57	3.20		
	Calc. $<\xi>$	1.15	1.86	(2.59)	(3.39)		

Table 4. Half-width of band.
t is the temperature in °C.
The values in parentheses are extrapolated values.

	s	
NaCl	1 / 1.5	(3.6)
KCl	1 / 1.4	(3.6)
RbCl	1 / 1.2	(3.6)
KBr	1 / 1.4	(3.7)

Table 5. Proper values for s.

The value of s is also obtainable from (3.7) which is independent of the value of σ'. Following this method we estimate $s=1/1.4$ for KBr from the data shown in Fig. 1. These orders of magnitude for s are reasonable and are of the same order as expected from the analysis of electrolytic conductivity of these crystals studied by Mott-Gurney.[7]

In conclusion we can say that the agreement with experiments is pretty good for calculated values in spite of its rough treatment. To avoid its descriptive nature of theory

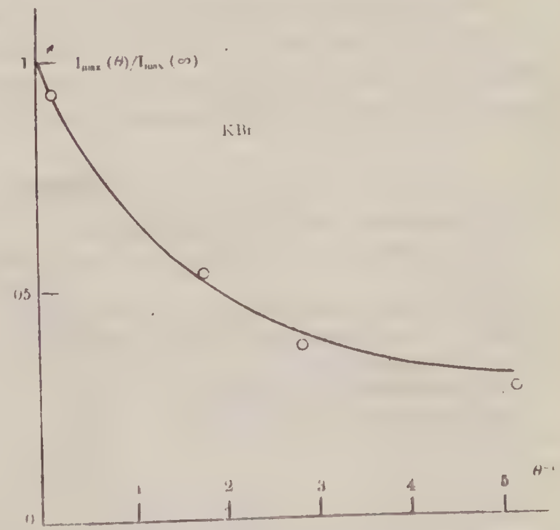

Fig. 5. The temperature dependence of peak intensity. The curve is the theoretical result calculated from (3.7).
The plotted points are reduced values from the data for KBr shown in Fig. 1.

we must deduce the value of factor g and s from the atomic point of view, but it is somewhat difficult because of the complex nature of polarization around the vacancy. Therefore we will postpone this problem till another time.

Summary.

Now we summarize our results.

1) The cause of the shift in the peak wave length with the temperature is mainly the result of the thermal expansion of the lattice. We can estimate the order of the magnitude of these effects and the results are shown in Fig. 3 and are in agreement with the experiments.
2) Using the approximation of the adiabatic potential we conclude that the width of the band corresponds to the variation of the energy gap in electronic levels caused by the vibration of the lattice. The intensity of absorption is proportional to the probability that the displacement of ions takes a value of corresponding wave length. This probability is the statistical average over several states of the vibration and the temperature effect of the band width is mainly determined by this mechanism. Assuming a simple model of the lattice vibration we can estimate the half-width of bands as a function of the temperature. Calculated values are in good agreement with the observed data over a wide range of temperature for various kinds of crystals. These results are shown in Fig. 4 and Fig. 5.

The authors are very grateful to Professor Kotani and Assistant Professor Kubo for their valuable discussions and thanks are due to Miss Mikoshiba for her assistance in the numerical work. The authors are indebted to the Japanese Council of Science for their financial help.

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On the Elimination of the Normal-dependent Part from the Hamiltonian

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§ 1. Introduction.

We know in the covariant formalism that the Hamiltonian density H depends on the normal of the space-like surface in many cases. It is because of the fact that the commutator of the Hamiltonians between two space-like points contains the 2nd order derivatives of the D-function, and the reason of the occurrence of such functions comes from that H contains the derivatives of the wave function of some field (meson field).

In this paper, we will see that it is possible to eliminate the normal dependent part from H by modifying Dyson's P-bracket.[1]

§ 2. The Normal-dependent Part of the Hamiltonian.

In the canonical formalism, we readily see the following propositions:

(I) The only field the derivatives of whose wave function is contained in the interaction Lagrangian density L is the meson field.

(II) In the above case, the derivatives are contained in L in the 1st order and linearly.

Then we can determine H from L readily. H is decomposed into normal-independent and -dependent parts;

$$H(x)=H_I(x)+H_{II}(x), \tag{2.1}$$

$$H_I(x)=-L(x),\ H_{II}(x)=K_{\mu\nu}(x)N_\mu N_\nu, \tag{2.2}$$

where N is the unit normal of the surface σ at the point x, and $K_{\mu\nu}$ is a tensor to be determined.

Now consider the integrability condition:

$$\left[\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)}+H(x),\ \frac{\hbar}{i}\frac{\delta}{\delta\sigma(x')}+H(x')\right]=0, \tag{2.3}$$

then so long as the propositions (I) and (II) hold, we can assume that

$$[H(x),H(x')]=[H_I(x),H_I(x')],$$

$$[H_I(x),H_{II}(x')]=[H_{II}(x),H_I(x')]=[H_{II}(x),H_{II}(x')]=0. \tag{2.4}$$

So the condition (2.3) reduces to the form

$$\frac{\hbar}{i}\left(\frac{\delta H_{II}(x')}{\delta\sigma(x)}-\frac{\delta H_{II}(x)}{\delta\sigma(x')}\right)+[H_I(x),H_I(x')]=0. \tag{2.5}$$

By the propositions (I) and (II), H_I is given by

$$H_I=K+K_\mu^*\frac{\partial\phi}{\partial x_\mu}+K_\mu\frac{\partial\phi^*}{\partial x_\mu}+K_\mu^0\frac{\partial\phi^0}{\partial x_\mu} \tag{2.6}$$

where K, K_μ^*, K_μ and K_μ^0 do not contain the derivative of the meson field wave functions and ϕ, ϕ^* and ϕ^0 refer to the charged (pseudo-)scalar and neutral (pseudo-) scalar meson fields respectively

Then the commutator of $H_I(x)$ and $H_{II}(x)$ is given by

$$\begin{aligned}[H_I(x),H_I(x')]&=K_\mu^*K_\nu'\left[\frac{\partial\phi}{\partial x_\mu},\frac{\partial\phi'^*}{\partial x_\nu'}\right]+K_\mu K_\nu'^*\left[\frac{\partial\phi^*}{\partial x_\mu},\frac{\partial\phi'}{\partial x_\nu'}\right]\\&\quad+K_\mu^0K_\nu'^0\left[\frac{\partial\phi^0}{\partial x_\mu},\frac{\partial\phi'^0}{\partial x_\nu'}\right]\\&=\frac{\hbar}{i}[K_\mu^*K_\nu'+K_\mu K_\nu'^*]\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'}+\frac{\hbar}{i}K_\mu^0K_\nu'^0\frac{\partial^2\Delta^0(x-x')}{\partial x_\mu\partial x_\nu'}\\&=\frac{1}{2}\cdot\frac{\hbar}{i}[K_\mu^*K_\nu+K_\mu K_\nu^*]\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'}+\frac{1}{2}\frac{\hbar}{i}K_\mu^0K_\nu^0\frac{\partial^2\Delta^0(x-x')}{\partial x_\mu\partial x_\nu'}\\&\quad-(\text{exchange terms of } x \text{ and } x'),\end{aligned} \tag{2.7}$$

where we employed Heaviside unit for the meson field, and put $c=1$, and Δ, Δ^0 the D-functions of the charged and the neutral mesons respectively. Combining (2.5) and (2.7), we get

$$\frac{\delta H_{II}(x)}{\delta\sigma(x')}=\frac{1}{2}(K_\mu^*K_\nu+K_\mu K_\nu^*)\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'}+\frac{1}{2}K_\mu^0K_\nu^0\frac{\partial^2\Delta^0(x-x')}{\partial x_\mu\partial x_\nu'}. \tag{2.8}$$

Integrating (2.8), we obtain

$$H_{II}=\frac{1}{2}(K_\mu^*K_\nu+K_\mu K_\nu^*+K_\mu^0K_\nu^0)N_\mu N_\nu. \tag{2.9}$$

From (2.6), K's are determined as

$$K_\mu^*=\frac{\partial H_I}{\partial\left(\frac{\partial\phi}{\partial x_\mu}\right)},\quad K_\mu=\frac{\partial H_I}{\partial\left(\frac{\partial\phi^*}{\partial x_\mu}\right)},\quad K_\mu^0=\frac{\partial H_I}{\partial\left(\frac{\partial\phi^0}{\partial x_\mu}\right)}. \tag{2.10}$$

So consequently, the total Hamiltonian H is given by

$$H=H_I+\frac{1}{2}\left[\frac{\partial H_I}{\partial\left(\frac{\partial\phi}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial\phi^*}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial\phi^*}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial\phi}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial\phi^0}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial\phi^0}{\partial x_\nu}\right)}\right]N_\mu N_\nu \tag{2.11}$$

If moreover, there exists (pseudo-)vector meson field, we must add the following terms in the bracket of (2.11);

$$\frac{\partial H_I}{\partial\left(\frac{\partial A_\sigma}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial A^*_\sigma}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial A^*_\sigma}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial A_\sigma}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial B}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial B^*}{\partial x_\nu}\right)}$$

$$+\frac{\partial H_I}{\partial\left(\frac{\partial B^*}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial B}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial A^0_\sigma}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial A^0_\sigma}{\partial x_\nu}\right)}+\frac{\partial H_I}{\partial\left(\frac{\partial B^0}{\partial x_\mu}\right)}\cdot\frac{\partial H_I}{\partial\left(\frac{\partial B^0}{\partial x_\nu}\right)}. \qquad (2.11')$$

We used Stueckelberg's form for the vector meson field in (2.11′). While H_I is the negative of L, so the formula (2.11) enables us to calculate H fiom L auto matically.

§ 3. Modification of Dyson's P-Bracket.

Let us consider Tomonaga-Schwinger equation:

$$\left\{\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)}+H(x)\right\}U[\sigma,\sigma_0]=0, \qquad (3.1)$$

with the initial condition $U[\sigma_0,\sigma_0]=1$.
Then, as is well known, the solution of the equation (3.1) is given by

$$U[\sigma,\sigma_0]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP(H(x_1),H(x_2),\cdots,H(x_n)) \; .(3.2)$$

We construct a family of space-like surfaces from σ_0 to σ, and we write $\sigma_1>\sigma_2$ to express that σ_1 lies in the future of σ_2.

The P-bracket in the integrand of (3.2) is defined by

$$P(H(x_1)H(x_2)\cdots H(x_n))=H(x_{i_1})H(x_{i_2})\cdots H(x_{i_n})$$

for 0) $\quad\sigma(x_{i_1})>\sigma(x_{i_2})>\cdots>\sigma(x_{i_n})$,

where $x_{i_1}, x_{i_2},\cdots$ are the permutation of $x_1, x_2,\cdots$.

But how shall we define it, if some of the surfaces $\sigma(x_1)$, $\sigma(x_2),\cdots$ coincide with each other? For instance,

i) $\quad\sigma(x_{i_1})=\sigma(x_{i_2})>\sigma(x_{i_3})>\cdots>\sigma(x_{i_n})$,

ii) $\quad\sigma(x_{i_1})=\sigma(x_{i_2})=\sigma(x_{i_3})>\cdots>\sigma(x_{i_n})$,

. . . .

In the ordinary theory, it was not necessary to define P for these cases.

Because $P(H(x_1)H(x_2)\cdots H(x_n))$ is the integrand of $4n$-dimensional integral, while the domain of the case i) is $(4n-1)$-dimensional, and $(4n-2)$-dimensional for ii), and so on, i. e., the measure of the domains of i), ii),$\cdots$ is zero, so that these cases do not contribute to the integral.

In this paper, we will admit such a discussion for the cases ii), iii),$\cdots$, but will consider more deeply about the case i), then P is decomposed into

$$P(H(x_1)\cdots H(x_n))=P(H(x_{i_1})H(x_{i_2}))P(H(x_{i_3})\cdots H(x_{i_n})).$$

Our probrem is to define $P(H(x_{i_1})H(x_{i_2}))$, while generally the following formula is valid:

$$P(H(x)H(x'))=\frac{1}{2}\{H(x),H(x')\}+\frac{1}{2}\varepsilon(x,x')[H(x),H(x')], \quad (3.3)$$

where
$$\varepsilon(x,x')=\begin{cases}1, & \text{for } \sigma(x)>\sigma(x').\\ -1, & \text{for } \sigma(x')>\sigma(x).\end{cases}$$

The case when $\varepsilon(x,x')$ troubles us is $\sigma(x)=\sigma(x')$, because in this case ε is not defined. If $[H(x),H(x')]=0$, the indefiniteness of $\varepsilon(x,x')$ does not trouble us, but if $[H(x),H(x')]\neq 0$, then it will be of the following form as discussed in §2:

$$[H(x),H(x')]=2\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\varDelta(x-x')}{\partial x_\mu\partial x'_\nu}. \quad (3.4)$$

Consequently $P(H(x)H(x'))$ is written as

$$P(H(x)H(x'))=\frac{1}{2}\{H(x),H(x')\}+\varepsilon(x,x')\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\varDelta(x-x')}{\partial x_\mu\partial x'_\nu}. \quad (3.5)$$

Here, in order to avoid the indefiniteness of P, we define the modified P-bracket P^* as follows:

$$P^*(H(x)H(x'))=\frac{1}{2}\{H(x),H(x')\}+\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\bar{\varDelta}(x-x')}{\partial x_\mu\partial x'_\nu}, \quad (3.6)$$

then $\bar{\varDelta}(x-x')=\varepsilon(x,x')\varDelta(x-x')$ does not depend on the family, and P^* coincides with P for $\sigma(x)\neq\sigma(x')$.

The general definition of P^* is:

case 0) $P^*(H(x_1)H(x_2)\cdots H(x_n))=P(H(x_1)H(x_2)\cdots H(x_n))$.

case i) $P^*(H(x_1)H(x_2)\cdots H(x_n))=P^*(H(x_{i_1})H(x_{i_2}))P(H(x_{i_3})\cdots (x_{i_n}))$.

For the cases ii), iii),⋯, P^* is left undefined, because as mentioned before, they do not contribute to the integral.

Using the new P^* bracket, let us consider the following operator

$$U^*[\sigma,\sigma_0]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_1)H_I(x_2)\cdots H_I(x_n)) \quad (3.7)$$

$$=\sum_{n=0}^{\infty}U_n[\sigma,\sigma_0]. \quad (3.8)$$

Now suppose that the surface σ is deformed by an infinitesimal volume $d\omega$ at a point x, and becomes σ', then

$$U_n[\sigma',\sigma]-U_n[\sigma,\sigma_0]=\binom{n}{1}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{\sigma}^{\sigma'}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_1)\cdots H_I(x_n))$$

$$+\binom{n}{2}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2\int_{\sigma_0}^{\sigma}dx_3\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_1)\cdots H_I(x_n))$$

$+\cdots\cdots$

If the integral $\int_{\sigma_1}^{\sigma_2}$ is defined for the domain $[\sigma_1,\sigma_2)$, then $[\sigma_0,\sigma)$ and $[\sigma,\sigma')$ have no common point, so

$$=\left(\frac{-i}{\hbar}\int_\sigma^{\sigma'}H_I(x)dx\right)\left[\frac{1}{(n-1)!}\left(\frac{-i}{\hbar}\right)^{n-1}\int_{\sigma_0}^{\sigma}dx_2\int_{\sigma_0}^{\sigma}dx_3\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_2)\cdots H_I(x_n))\right]$$

$$+\left[\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2P^*(H_I(x_1)H_I(x_2))\right]$$

$$\times\left[\frac{1}{(n-2)!}\left(\frac{-i}{\hbar}\right)^{n-2}\int_{\sigma_0}^{\sigma}dx_3\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_3)\cdots H_I(x_n))\right]$$

$+\cdots\cdots$

$$=\left(\frac{-i}{\hbar}\int_\sigma^{\sigma'}H_I(x)dx\right)U_{n-1}[\sigma,\sigma_0]+\left[\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2P^*(H_I(x_1)H_I(x_2))\right]$$

$$\times U_{n-2}[\sigma_1,\sigma^0]+\cdots$$

In the limit $d\omega\to 0$,

$$\lim_{d\omega\to 0}\frac{1}{d\omega}\left(\frac{-i}{\hbar}\int_\sigma^{\sigma'}H_I(x)dx\right)=-\frac{i}{\hbar}H_I(x),$$

$$\lim_{d\omega\to 0}\frac{1}{d\omega}\left[\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int_\sigma^{\sigma'}dx_1\int_\sigma^{\sigma'}dx_2P^*(H_I(x_1)H_I(x_2))\right]\equiv-\frac{i}{\hbar}\bar{H}(x), \tag{3.9}$$

other terms vanish by the reason mentioned before, and therefore we get

$$i\hbar\frac{\delta}{\delta\sigma(x)}U_n[\sigma,\sigma_0]=H_I(x)U_{n-1}[\sigma,\sigma_0]+\bar{H}(x)U_{n-2}[\sigma,\sigma_0]. \tag{3.10}$$

Now our task is to evaluate $\bar{H}(x)$, and in order that $\bar{H}(x)$ does not vanish, it is necessary that $P^*(H_I(x)H_I(x'))$ contains 4-dimensional δ-function.

$\bar{H}(x)$ is given by

$$\bar{H}(x)=\frac{1}{2}\left(\frac{-i}{\hbar}\right)\lim_{d\omega\to 0}\frac{1}{d\omega}\int_\sigma^{\sigma'}dx\int_\sigma^{\sigma'}dx'P^*(H_I(x)H_I(x')) \tag{3.11}$$

while P^* is

$$P^*(H_I(x)H_I(x'))=\frac{1}{2}\{H_I(x)H_I(x')\}+\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'}, \tag{3.12}$$

where $K_{\mu\nu}(x,x)=K_{\mu\nu}(x)$.

The evaluation of $\{\ \}$ is performed readily.

$$\lim_{d\omega\to 0}\frac{1}{d\omega}\cdot\frac{1}{2}\int_\sigma^{\sigma'}dx\int_\sigma^{\sigma'}dx'\{H_I(x)H_I(x')\}=\lim_{d\omega\to 0}\frac{1}{d\omega}\left[\int_\sigma^{\sigma'}dx\,H_I(x)\right]^2=0.$$

The evaluation of the rest term is given by

$$\bar{H}(x)=-\frac{1}{2}\lim_{d\omega\to 0}\frac{1}{d\omega}\int_{\sigma}^{\sigma'}dx\int_{\sigma}^{\sigma'}dx'K_{\mu\nu}(x,x')\frac{\partial^2\bar{\Delta}(x-x')}{\partial x_\mu\partial x_\nu'}$$

$$=-\frac{1}{2}\lim_{d\omega\to 0}\int_{\sigma}^{\sigma'}dx'K_{\mu\nu}(x,x')\frac{\partial^2\bar{\Delta}(x-x')}{\partial x_\mu\partial x_\nu'}. \qquad (3.1\ )$$

If we choose a Lorentz system whose time axis coincides with the normal of σ at x, then for a space-like point x'

$$\frac{\partial^2\bar{\Delta}(x-x')}{\partial^2 x_{\bar{4}}\partial x_{\bar{4}}}=2\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')\delta(x_0-x_0'). \qquad (3.14)$$

Inserting (3.14) into (3.13), one obtains easily

$$\bar{H}(x)=K_{\bar{4}\bar{4}}(x,x)=K_{\mu\nu}(x,x)N_\mu N_\nu=K_{\mu\nu}(x)N_\mu N_\nu$$

$$=H_{II}(x). \qquad (3.15)$$

Again inserting this result into (3.10), and summing up (3.10) with respect to n, we get

$$\left\{\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)}+H(x)\right\}U^*[\sigma,\sigma_0]=0. \qquad (3.16)$$

Clearly U and U^* satisfy the same equation (3.1) and (3.16), and the same initial condition $U[\sigma_0,\sigma_0]=U^*[\sigma_0,\sigma_0]=1$.

But as the Hamiltonian of this equation satisfies the integrability condition, the solution must be unique, thus we know

$$U^*[\sigma,\sigma_0]=U[\sigma,\sigma_0] \qquad (3.17)$$

i. e.

$$U[\sigma,\sigma_0]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(H_I(x_1)\cdots H_I(x_n))$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_nP^*(L(x_1)\cdots L(x_n)). \qquad (3.18)$$

If we calculate S-matrix, then it is given by

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\cdots\int_{-\infty}^{\infty}dx_nP^*(L(x_1)\cdots L(x_n)). \qquad (3\cdot 19)$$

This expression coincides with that proposed by Koba,[2] but the definition of P^* is different.

A concrete example had been given by Matthews[3] for the case of interacting nucleon-scalar meson field with vector coupling, and the present theory is the extention of his treatment.

One will see, using this P^*-bracket, the following formulae:

$$< P(\phi^0(x), \phi^0(x')) >_0 = < P^*(\phi^0(x), \phi^0(x') >_0 = (\hbar/2)\Delta_F(x-x'),$$

$$< P\left(\frac{\partial\phi^0}{\partial x_\mu}, \frac{\partial\phi'^0}{\partial x'_\nu}\right) >_0 \neq < P^*\left(\frac{\partial\phi^0}{\partial x_\mu}, \frac{\partial\phi'^0}{\partial x'_\nu}\right) >_0 = \frac{\hbar}{2}\frac{\partial^2\Delta_F(x-x')}{\partial x_\mu \partial x'_\nu}.$$

As is seen in the above formulae, the P^*-bracket is simpler than the P-bracket when the derivatives of wave functions appear, and moreover the normal-dependent part is omitted from our calculation.

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The Mutual Repulsive Potential between Argon Atoms

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Abstract

Bleick and Mayer developed a method for treating the repulsive interaction of two closed shells, and determined the interaction of two neon atoms. In the present paper the same method was applied to calculate the repulsive potential in the case of two argon atoms, the one-electron atomic wave functions that were obtained numerically by Hartree being used. The value of the repulsive potential at $R=3.70 \times 10^{-8}$cm is 6.20×10^{-14} ergs; at $R=2.64 \times 10^{-8}$cm is 328×10^{-14} ergs; and at $R=2.12 \times 10^{-8}$cm is 2090×10^{-14} ergs. These three values are fitted by the simple function $b \exp(-R/\rho)$ with small error as expected.

Introduction

Theoretical derivations of the thermodynamical quantities from the intermolecular potentials have been made largely for the case of rare gases, because the molecules of rare gases are monatomic and chemically saturated, and the intermolecular potential of a pair of these molecules depends only on the intermolecular distance.

Out of these rare gases, argon is suitable to carry out various experiments at the ordinary temperature, and there are a number of experimental data on the properties of argon available for the comparison with theory.

The interaction potential curves for rare gas atoms have been investigated by many workers. Lennard-Jones[1] and Buckingham[2] assumed an interaction energy of the form

$$\Delta E(R) = -AR^{-6} + BR^{-n}. \tag{1}$$

And Buckingham assumed the form

$$\Delta E(R) = -aR^{-6} + b \exp(-R/\rho). \tag{2}$$

The values of constants A and B or a, b and ρ were determined so as to make the temperature dependence of the second virial coefficient calculated from these interatomic potential fit as accurately as possible; the evaluation of A and B was made for $n=8$, 9, 10, 12, 14. And Rice[3] determined the more complicated form of the interatomic potential of a pair of argon atoms from the specific heat, the thermal expansion coefficient, the binding energy and the interatomic distance in solid argon besides the second virial coefficient.

To determine the interatomic potential function of argon, we need not, in principle, use the experimental data mentioned above, since it can be determined by quantum mechanical calculation. And Bleick and Mayer[4] have developed an excellent method for calculating the mutual repulsive interaction of two ions or atoms having a closed rare gas configuration, and they determined the interaction energy in the case of two neon atoms, using one-electron atomic functions that were obtained by approximating Brown's results analytically. This method is essentially equivalent to the method of Heitler and London[5]. The purpose of the present paper is to calculate the mutual repulsive potential energy of two argon atoms by Bleick and Mayer's method.

The Theory

The potential is determined as a perturbation energy of two atoms in a manner exactly analogous to the method of Heitler and London[5]. Two argon atoms a, and b of nuclear charge Z_a and Z_b have a nuclear distance R. As there are eighteen electrons in each atom, we number them from 1 to 36. r_{ak} and r_{bk} denote the distances of electron k to the nuclei a and b respectively. The zeroth order wave function for the system of two argon atoms are expressed by a linear combination of permutations of the product of one-electron hydrogenlike functions,

$$\Psi_0=(36!)^{-\frac{1}{2}}\sum_P(-1)^p\Psi_P, \qquad \Psi_P=P(\Psi_1),$$

$$\Psi_1=\psi_{a1s}\alpha(1)\cdot\psi_{a1s}\beta(2)\cdot\psi_{a2s}\alpha(3)\cdots\psi_{a3p-}\beta(18)\psi_{b1s}\alpha(19)\cdots\psi_{b3p-}\beta(36), \tag{3}$$

where ψ_{a1s}, ψ_{a2p0}, ψ_{a3p-} are one-electron (Hartree) functions of atom a, of $1s$, of $2p$ with $m=0$ and of $3p$ with $m=-1$ respectively, and α and β represent the two possible spin functions. Here P is a permutation, and p has the parity of P.

The Hamiltonian operator for 36 electrons is written in the form,

$$H=-\sum_{k=1}^{36}\frac{\hbar^2}{2m}\Delta_k+e^2\left[\frac{Z_aZ_b}{R}-\sum_k^{36}\left(\frac{Z_a}{r_{ak}}+\frac{Z_b}{r_{bk}}\right)+\sum_{k=1}^{36}{}'\sum_{l=1}^{36}\frac{1}{r_{kl}}\right], \tag{4}$$

where $-\frac{\hbar^2}{2m}\Delta_k$ is the kinetic energy of the kth electron. It should be noted that the case $k=l$ are excluded in the last summation. Then, if $E_{0a}+E_{0b}$ is the sum of the energies of the two isolated atoms and Ψ is the complete electronic wave function for 36 electrons of the two argon atoms, the interaction energy of the two argon atoms, ΔE, is

$$\Delta E=\int\Psi^*H\Psi d\tau-(E_{0a}+E_{0b}), \tag{5}$$

where the integral means integration and summation over the positional and the spin coordinates respectively. We shall develop this in the case where Ψ is expressed in terms of the solutions of Hartree's[6] or Fock's[7] equations. Replacing $E_{0a}+E_{0b}$ by the sum of the energies in the Hartree or Fock approximation, and using the one electron wave functions, we obtain, from Eq. (5),

$$(1-S)\Delta E=\frac{1}{36!}\sum_{P}\sum_{P'}(-1)^{p+p'}\int\Psi_P\Delta V_{P'}\Psi_{P'}d\tau, \tag{6}$$

where

$$\Delta V_P=P(\Delta V_1), \tag{7}$$

$$\Delta V_1=e^2\left[\frac{Z_aZ_b}{R}-\sum_{k=1}^{18}\frac{Z_b}{r_{bk}}-\sum_{l=19}^{36}\frac{Z_a}{r_{al}}+\sum_{k=1}^{18}\sum_{l=19}^{36}\frac{1}{r_{kl}}\right]. \tag{8}$$

S corresponds to the S of Heitler and London and has a small value (~ 0.01):

$$-S=\frac{1}{36!}\sum_{P}{}'\sum_{P'}(-1)^{p+p'}\int\Psi_P^*\Psi_{P'}d\tau, \tag{9}$$

in which $\sum'$ indicates that P and P' never takes identical values. As will become evident later, only permutation of one pair needs be considered, and Eq. (9) may be rewritten in the form

$$S=2\sum_{ij}S_{ij} \qquad (i, j=1s, 2s, 3s, 2p_0, \cdots 3p_-),$$
$$S_{ij}=|s_{ij}|^2, \qquad s_{ij}=\int\psi_{ai}^*\psi_{bj}d\tau. \tag{10}$$

The numbers of electron in the integrand of Eq. (6) are immaterial for the numerical value of the integral. We, therefore, change the notation of electrons in each term taken as a whole, so that $\Psi_{P'}$ and $\Delta V_{P'}$ become Ψ_1 and ΔV_1, obtaining the 36! different terms instead of $(36!)^2$ terms in Eq. (6). This gives

$$\sum_{P}^{36!\ \text{terms}}(-1)^p\int\Psi_P^*\Delta V_1\Psi_1 d\tau=\Delta E(1-S). \tag{11}$$

As shown by Bleick and Mayer, out of the 36! terms in Eq. (11), only several terms are different from each other and appreciably different from zero.

Eq. (11) can be simplified by the following reasons:

i) ΔV_1 is independent of the spin coordinates and α and β are orthogonal.

ii) The perturbation potential contains no terms with the coordinates of more than one electron on the same atom and $\int\psi_{ai}^*\psi_{aj}d\tau=\delta_{ij}$.

iii) The perturbation potential contains no terms with the coordinates of more than two electrons, and $\int\psi_{ai}^*\psi_{bj}d\tau=s_{ij}\ll 1$. It enables us to neglect the terms in which more than one pair of electrons has different positions in Ψ_P^* and Ψ_1.

In order to simplify the integrals, we will introduce new functions ρ_a, ρ_b, ρ'_{ai}, ρ'_{bj}, U_a, U_b, U'_{ai}, and U'_{bj} which are the functions of the distance from nuclei a or b.

ρ_a is the density of electric charge (in units of e) in the atom a. ρ'_{ai} is the density of charge in the atom a with the ith electron missing. U_a is the potential due to atom a. U'_{ai} is the potential due to a when ith electron is missing. Strictly speaking, ρ'_{ai} or U'_{ai} depends also on φ_a and θ_a, but we neglect the angular parts as in the calculations of the Hartree functions.

These new functions are expressed as

$$\rho_a(r_a)=-\sum|\psi_{ai}|^2+Z_a\delta(r_a)/4\pi r_a^2, \tag{12}$$

$$\rho'_{ai}(r_a) = \rho_a + |R_{ai}(r_a)|^2/4\pi, \tag{13}$$

where $\delta(r_a)$ is Dirac's delta-function, and $R_{ai}(r_a)$ is the radial function (normalized) of the ith electron of the atom a.

$$U_a(r_a) = \frac{1}{r_a}\int_0^{r_a} 4\pi r^2 \rho_a(r)dr + \int_{r_a}^{\infty} 4\pi r \rho_a(r)dr, \tag{14}$$

which is identical with

$$U_a(r_a) = \frac{Z_a}{r_a} - \sum_{\text{all electrons}} \int \psi_{ai}^* \frac{1}{r_i} \psi_{ai} d\tau, \tag{15}$$

where r_i is the distance from some point with the coordinate r_a to the ith electron. The potential $U'_{ai}(r_a)$ is given by Eq. (14) in which ρ'_{ai} is substituted for ρ_a.

Using these functions, Eq. (11) is transformed after rather complicated consideration, and becomes

$$\Delta E = W_{11} - [2/(1-S)]\sum_{ij}(s_{ij}U'_{ij} + R_{ij}^{-1}),$$

$$(i, j = 1s, 2s, 3s, 2p_+, 2p_0, 2p_-, 3p_+, 3p_0, 3p_-), \tag{16}$$

$$W_{11} = e^2 \int U_a \rho_b d\tau = e^2 \int U_b \rho_a d\tau, \tag{17}$$

$$U'_{ij} = -e^2 \int (U'_{ai} + U'_{bj}) \psi_{ai}^* \psi_{bj} d\tau, \tag{18}$$

$$R_{ij}^{-1} = e^2 \int\int \psi_{ai}^* \psi_{bj}(1)(1/r_{12}) \psi_{bj}^* \psi_{ai}(2) d\tau_1 d\tau_2. \tag{19}$$

The Calculations

The solutions of Fock's equations for argon atom has been determined by Hartree and Hartree[8] numerically. Using these functions, we calculate the integrals (17)–(19) by means of numerical integration. The one-electron hydrogen-like function in Eq. (3) is written in the form

$$\psi_{nlm} = \frac{f_{nl}}{r} S_{lm}(\theta, \varphi),$$

where S_{lm} is a normalized spherical harmonic and $f_{nl}(r)/r$ is the radial part of the wave function. The functions appeared in the previous Section can be all derived from $f_{nl}(r)$. In the case of argon, the quantum numbers of the states nl are $1s$, $2s$, $3s$, $2p$ and $3p$.

$\rho_a(r_a)$ in Eq. (12) is calculated from

$$\rho_a(r_a) = -\sum \frac{2l+1}{2\pi} \frac{f_{nl}^2}{r_a^2} + Z_a \frac{\delta(r_a)}{4\pi r_a^2}, \qquad (Z_a = 18),$$

and $\rho'_{ai}(r_a)$ from

$$\rho'_{ai}(r_a) = \rho_a(r_a) + f_{nl}^2/4\pi r_a^2$$

and U_a and U'_{ai} can be calculated from $\rho_a(r_a)$ and $\rho'_{ai}(r_a)$, by means of Eq. (14).

To transform the integrals of Eqs. (16)–(19) into the form convenient for calculation, the method appeared in Landshoff's paper[7] on the lattice energy of NaCl lattice will be often utilised.

a. The Calculations of W_{11}

In the calculation of the integrals W_{11} given by Eq.(17), we must pay attention to the fact that $U_a(r_a)$ is a function of r_a and $\rho_b(r_b)$ is of r_b. Accordingly we must transform the coordinate before carrying out the integration. $U_a(r)$ is expanded in Legendre series (See Appendix A). For example, any function of distance R from one origin O, $F(R)$, is expanded as

$$F(R)=\rho_0(r)+\rho_1(r)P_1(\cos\theta)+\rho_2(r)P_2(\cos\theta)+\cdots \tag{20}$$

where r is the distance of the point to another origin O' separated from O by a distance a, θ is the angle between vector $\overrightarrow{r}$ and the line OO' and $P_n(\cos\theta)$ is the Legendre function of degree n. The coefficient $\rho_n(r)$ is given in Appendix A. $\rho_0(r)$ is given by

$$\rho_0(r)=\frac{1}{2}\int_0^\pi F(R)\sin\theta\,d\theta=\frac{1}{2ar}\int_{|a-r|}^{a+r}F(R)R\,dR. \tag{21}$$

Now, expanding U_a by Eq. (20), we can write Eq. (17) in the form

$$W_{11}=e^2\int(\rho_0(r)+\rho_1(r)P_1(\cos\theta)+\cdots)\rho_b r^2\sin\theta d\theta dr d\varphi.$$

Taking into account the orthogonality of Legendre function, we have

$$W_{11}=e^2\int\rho_0\rho_b r^2\sin\theta d\theta dr d\varphi=4\pi e^2\int\rho_0\rho_b r^2 dr. \tag{22}$$

The results are tabulated in Table I.

Table I The Values of W_{11}

	I	II	III	IV	V
a (Bohr Units)	7	5	4	3	2
W_{11} (10^{-14} ergs)	0.98	44.0	410	2420	10700

b. The Calculations of U'_{ij}

U'_{ij} given by Eq. (18) can be written in the form

$$U'_{ij}=-e^2\int U'_{ai}\psi^*_{ai}\psi_{bj}d\tau-e^2\int U'_{bj}\psi^*_{ai}\psi_{bj}d\tau.$$

In the case of two argon atoms, the functional forms of U'_{ai} and U'_{bi} are identical. Then we may interchange the role of the suffices a and b in the first term of the above equation, obtaining

$$\begin{aligned}U'_{ij}&=-e^2\int U'_{bi}\psi^*_{bi}\psi_{aj}d\tau-e^2\int U'_{bj}\psi^*_{ai}\psi_{bj}d\tau\\&=-e^2\int(\psi_{aj}U'_{bi}\psi^*_{bi}+\psi^*_{ai}U'_{bj}\psi_{bj})d\tau.\end{aligned} \tag{23}$$

The states i and j are expressed by the quantum numbers nlm. The magnetic quantum number m of p-states takes the value 1, 0 and -1. Combining the three wave functions for the different values of m properly, we obtain three independet wave functions $\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{f_{nl}(r)}{r}\cos\theta$, $\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{f_{nl}(r)}{r}\sin\theta\sin\varphi$, $\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{f_{nl}(r)}{r}\sin\theta\cos\varphi$. We designate them as 0, sin and cos respectively and we will, hereafter, use the notations $1s$, $2s$, $2p_0$, $2p_{\sin}$, $2p_{\cos}$, $3s$, $3p_0$, $3p_{\sin}$ and $3p_{\cos}$.

In the calculation of Eq. (23), ψ_{ai}'s are transformed as shown in Appendix B, and integrals are calculated in a manner as described in Appendix C.

The results are tabulated in Table II.

c. The Calculations of s_{ij}.

$s_{ij}=\int\psi_{ai}^*\psi_{bj}d\tau$ is obtained at once from Appendix C. From the numerical value of s_{ij}, it was found that the method of approximations used here is not legitimated at the short nuclear separation such as $a=2$ and 3 Bohr Units. Hence, the calculations were carried out only for the nuclear separation $a=4$, 5 and 7 Bohr Units.

Thus the results are given in the third column of Table II.

d. The Calculation of R_{ij}^{-1}

$\sum_{ij}R_{ij}^{-1}$ given by Eq. (19) can be written in the form

$$\sum_{ij}R_{ij}^{-1}=e^2\sum_{ij}\iint\psi_{ai}^*(1)\psi_{ai}^*(2)r_{12}^{-1}\psi_{bj}(1)\psi_{bj}(2)d\tau_1d\tau_2.$$

Let us replace the suffix j with nlm and change the position of indices a and b as shown in Appendices B and C. Then we have

$$\sum_{ij}R_{ij}^{-1}=e^2\sum_{ij}\iint\psi_i^a(1)\psi_i^a(2)r_{12}^{-1}\psi_{nlm}^b(1)\psi_{nlm}^b(2)d\tau_1d\tau_2$$

$$=e^2\sum_i\iint\frac{\psi_i^a(1)\psi_i^a(2)\sum_{nl}\rho_{nl}^b(12)}{r_{12}}d\tau_1d\tau_2, \tag{24}$$

where

$$\rho_{nl}^b(12)=\sum_m\psi_{nlm}^b(1)\psi_{nlm}^b(2)=\frac{2l+1}{4\pi}\frac{f_{nl}(1)f_{nl}(2)}{r_1r_2}P_l(\cos u). \tag{25}$$

u being the angle between the vectors $\vec{r_1}$ and $\vec{r_2}$. In Eq. (24) the term r_{12}^{-1} or $\cos u/r_{12}$ appears. According to the properties of the Legendre polynomials we have

$$\frac{1}{r_{12}}=\sum_h\frac{r_i^h}{r_k^{h+1}}P_h(\cos u) \tag{26}$$

Table II Numerical Values of The Integrals for Two Argon Atoms
($3p_1$ is the mean of $3p_{\cos}$ or $3p_{\sin}$, and $2p_1$ is the mean of $2p_{\cos}$ or $2p_{\sin}$)

	R	i	j	$s_{ij}(10^{-2})$	$S_{ij}(10^{-4})$	$-\frac{U_{ij}'}{e^2}(10^{-2})$	$-\frac{s_{ij}U_{ij}'}{e^2}(10^{-4})$
		$3p_0$	$3p_0$	2.88	8.29	1.82	5.24
		$3s$	$3p_0$	−1.17	1.37	−1.27	1.49
		$3p_1$	$3p_1$	0.488	0.238	0.376	0.183
	7 B. U.	$3s$	$3s$	0.255	0.650	0.228	0.0581
I		$2s$	$3p_0$	0.085	0.0072	0.1	0.009
	3.70 A	$2p_0$	$3p_0$	−0.0402	0.0016	−0.1	0.004
		$2s$	$3s$	−0.00732		−0.002	
		$3s$	$2p_0$	0.0074		0.001	
		$3p_0$	$3p_0$	13.3	177	17.7	235
		$3s$	$3p_0$	−7.66	58.7	−10.4	79.7
		$3p_1$	$3p_1$	3.42	11.7	3.28	11.2
	5 B. U.	$3s$	$3s$	2.90	8.41	3.4	9.9
II		$2s$	$3p_0$	0.844	0.712	0.86	0.726
	2.64 A	$2p_1$	$3p_0$	−0.372	0.138	−0.6	0.222
		$2s$	$3s$	−0.145	0.0210	−0.28	0.040
		$3s$	$2p_0$	−0.0275	0.000756	0.22	0.006
		$3p_0$	$3p_0$	23.5	552	35.4	832
		$3s$	$3p_0$	−18.6	346	−24.8	461
		$3p_1$	$3p_1$	8.96	80.3	9.76	87.4
	4 B. U.	$3s$	$3s$	8.77	76.9	12.1	106
III		$2s$	$3p_0$	2.58	6.66	30	77
	2.12 A	$2p_0$	$3p_0$	−1.06	1.12	−8.0	8.5
		$2s$	$3s$	−0.66	0.44	−8.9	5.9
		$2p_1$	$3s$	0.41	0.168	2.6	1.1

and

$$\frac{\cos\alpha}{r_{12}}=\sum_{\lambda}\left(\frac{\lambda}{2\lambda-1}\frac{r_i^{\lambda-1}}{r_k^{\lambda}}+\frac{\lambda+1}{2\lambda+3}\frac{r_i^{\lambda+1}}{r_k^{\lambda+2}}\right)P_\lambda(\cos\alpha) \tag{27}$$

where $r_i=r_1$, $r_k=r_2$ when $r_1<r_2$, and $r_i=r_2$, $r_k=r_1$ when $r_1>r_2$.

Furthermore we need the addition theorem for the Legendre polynomials

$$P_\lambda(\cos\alpha)=P_\lambda(\cos\theta_1)P_\lambda(\cos\theta_2)+2\sum_m\frac{(\lambda-m)!}{(\lambda+m)!}P_\lambda^m(\cos\theta_1)P_\lambda^m(\cos\theta_2)\cos m(\varphi_1-\varphi_2), \tag{28}$$

where $P_\lambda^m(\cos\theta)$ is an associated Legendre function of degree λ and order m; and also we use the formula which replace the double integral with twice integrals

$$\iint c(r_1)c(r_2)\,\frac{r_i^s}{r_k^{s+1}}dr_1dr_2=2\int_0^\infty\int_0^{r_2}c(r_1)r_1^s dr_1\,\frac{c(r_2)}{r_2^{s+1}}dr_2. \tag{29}$$

At the first place, we represent the function $\psi_i^p(1)$ and $\psi_i^q(2)$ in Eq. (24) as functions of the coordinates relative to a new origin O'. The results are given by Eqs. (B5), (B8) and (B9).

Now, we write the right-hand side of Eq. (24) as $\sum_{inl} C_{i,nl}$. And we consider the angular part of $C_{i,nl}$. $P_l(\cos\alpha)$ in $\rho_{nl}(12)$ given by Eq. (28) is equal to 1 for s-state and $\cos\alpha$ for p state, and $P_l(\cos\alpha)/r_{12}$ gives the result of Eq. (26) or

Eq. (27). $P_h(\cos\alpha)$ in Eq. (26) or Eq. (27) is expanded as Eq. (28). On the other hand from the factor given by Eqs. (B5), (B8) or (B9), w find the angular part to be $P_l(\cos\theta)$ in s or p_0-state, and $P^1(\cos\theta)\frac{\sin}{\cos}\varphi$ in $np_{\sin}$- and $p_{\cos}$ states. Multiplying these terms with Eq. (28) and integrating over θ and φ, we obtain, for the product of s or p_0-state

$$\int P_l(\cos\theta_1)P_{l'}(\cos\theta_2)P_h(\cos\alpha)\,d\Omega_1 d\Omega_2=\delta_{lh}\delta_{l'h}\frac{4\pi}{2l+1}\,\frac{4\pi}{2l'+1}=\frac{16\pi^2}{(2h+1)^2}$$

and, for the product of $p_{\sin}$-state and $p_{\sin}$-state, and for the product of $p_{\cos}$-state and $p_{\cos}$-state,

$$\int \frac{\sin}{\cos}\varphi_1\frac{\sin}{\cos}\varphi_2\, P_l^1(\cos\theta_1)P_{l'}^1(\cos\theta_2)P_h(\cos\alpha)\,d\Omega_1 d\Omega_2$$

$$=2\delta_{lh}\delta_{l'h}\frac{2\pi l(l+1)}{2l+1}\,\frac{2\pi l'(l'+1)}{2l'+1}\,\frac{1}{h(h+1)}=\frac{8\pi^2 l(l+1)}{(2h+1)}$$

And other products vanish owing to the integration over the variable φ. Thus we obtain the following results,

$$C_{ns,ns}=\sum_h\frac{2}{(2h+1)^2}\iint\rho_h^{ns}(r_1)f_{ns}(r_1)r_1^{h-1}dr_1\frac{\rho_h^{ns}(r_2)f_{ns}(r_2)}{r_2^{h+2}}\,dr_2, \tag{30}$$

$$C_{ns,np}=\sum_h\frac{6}{(2h+1)^2}\iint\rho_h^{ns}(r_1)f_{np}(r_1)\rho_h^{ns}(r_2)f_{np}(r_2)\left(\frac{h}{2h-1}\,\frac{r_1^{h-2}}{r_2^{h+1}}+\frac{h+1}{2h+3}\,\frac{r_1^h}{r_2^{h+3}}\right)dr_1dr_2, \tag{31}$$

$$C_{npo,ns}=\sum_h\frac{18a^2}{(2h+1)^2}\iint E_h(r_1)f_{ns}(r_1)r_1^{h-1}dr_1E_h(r_2)f_{ns}(r_2)r_2^{-h-2}dr_1dr_2, \tag{32}$$

$$C_{npo,np}=\sum_h\frac{18a^2}{(2h+1)^2}\iint E_h(r_1)f_{np}(r_1)E_h(r_2)f_{np}(r_2)\left(\frac{h}{2h-1}\,\frac{r_1^{h-2}}{r_2^{h+1}}+\frac{h+1}{2h+3}\,\frac{r_1^h}{r_2^{h+3}}\right)dr_1dr_2, \tag{33}$$

$$C_{np_{\sin},ns}=C_{np_{\cos},ns}=\sum_h\frac{3(h+1)(h+2)}{(2h+3)^2}\iint D_{h+1}(r_1)f_{ns}(r_1)r_1^h dr_1\frac{D_{h+1}(r_2)f_{ns}(r_2)}{r_2^{h+3}}\,dr_2, \tag{34}$$

$$C_{np_{\sin},np}=C_{np_{\cos},np}=\sum_h\frac{9(h+1)(h+2)}{(2h+3)^2}\iint D_{h+1}(r_1)f_{np}(r_1)D_{h+1}(r_2)f_{np}(r_2)$$

$$\times\left(\frac{h+1}{2h+1}\,\frac{r_1^{h-1}}{r_2^{h+2}}+\frac{h+2}{2h+5}\,\frac{r_1^{h+1}}{r_2^{h+4}}\right)dr_1dr_2, \tag{35}$$

where

$$aE_h(r)=-\frac{h}{h-1}r\rho_{h-1}^{np}(r)+a\rho_h^{np}(r)-\frac{h+1}{2h+3}r\rho_{h+1}^{np}(r),$$

$$aD_h(r)=\frac{1}{2h-1}\rho_{h-1}^{np}(r)-\frac{1}{2h+3}\rho_{h+1}^{np}(r).$$

And $\sum_{ij} R_{ij}^{-1}$ is expressed by

$$\sum R_{ij}^{-1} = 2(C_{2s,3p} + C_{3s,3p} + C_{2p_\sigma,3s} + C_{3p_\sigma,2s} + C_{2p_{\cos},3p} + 2C_{2p_{\cos},3s} + 2C_{3p_{\cos},2s} + C_{2s,3s}$$

$$+ C_{3p_{\cos},3p}) + C_{3s,3s} + C_{2p_\sigma,3p} + C_{3p_\sigma,3p}$$

where the terms containing the state of quantum number $n=1$ and the terms in which the both states belong to $n=2$ are negligible and omitted. The first three terms of the expansion of $C_{i,nl}$ are computed. And the results are tabulated in Table III.

e. Numerical Results

The numerical values of the integrals given by Eqs. (11) and (16)–(19) for two argon atoms are tabulated in Table IV. The equilibrium separation of argon lattice at the temperature of liquid helium is 3.82 Å.

Table III The Values of $C_{i,nl}$

i	nl	at 7 B. U.	at 5 B. U.	at 4 B. U.
$3p_1$	$3p$	0.342	15.70	105.1
$3p_0$	$3p$	0.0760	4.31	27.3
$3p_1$	$3s$	0.0356	2.31	19.4
$3s$	$3s$	0.027	4.15	41.7
$3p_0$	$3s$	0.024	3.11	23.3
$2s$	$3p$	0.0060	0.720	8.38
$2p_1$	$3p$	0.001	0.008	0.30
$2s$	$3s$		0.002	0.83
$2p_1$	$3s$			0.06

Table IV Sums

$R(10^{-8}$ cm)	S	$\frac{-2}{1-S}\Sigma s_{ij}U_{ij}$	$\frac{2}{1-S}\Sigma\frac{1}{R_{ij}}$	$-W_{11}$	ΔE
3.70	0.00232	7.63	0.447	0.98	6.20
2.64	0.0656	490	28.2	44.0	328
2.12	0.300	2750	271	410	2090

Discussion of Results

The empirical equations for the total potential of two like atoms are given in several forms, out of which the following two forms given by Eqs. (1) and (2) are often used:

$$\Delta E = -\frac{A}{R^6} + \frac{B}{R^n} \tag{36}$$

and

$$\Delta E = -aR^{-6} + b\exp(-R/\rho) \tag{37}$$

where A, B and n in Eq. (36) and a, b and ρ in Eq. (37) are constants. The first term $-AR^{-6}$ or $-aR^{-6}$ of the right hand side is the term for attraction, and the second for repulsion. Buckingham[2] gave the numerical values to all constants. As in our method only the repulsive term is deduced, we can determine the constants of repulsive term in Eq. (36) or in Eq. (37). The constants of repulsive term derived from Table IV are tabulated in Table V with the values of Buckingham and of Rushbrooke[11], method of least square being used to determine the constants.

Table V The Constants of the Potential Curve for Argon

$\Delta E = -AR^{-6} + BR^{-n}$

	n	A (10^{-10} ergsÅ^6)	B (10^{-10}ergsÅ^n)
i)	9	1.70	76.8
ii)	10	1.37	205
iii)	12	1.03	1620
iv)	11.4	1.12	869
v)	10.5		660
vi)	12		2970

$\Delta E = -aR^{-6} + b\exp(-R/\rho)$

	a (10^{-10} ergs Å^6)	ρ (Å)	b (10^{-10} ergs)
vii)	1.02	0.278	169
viii)		0.272	515

i, ii, iii, vii : by Buckingham,
iv, : by Rushbrooke
v, vi, viii : from the values obtained in this paper,
v : variating both n and B; vi : fixing $n=12$ and variating B.

In Table VI, the values obtained from the analytical forms Eq. (31) and Eq. (32) (the constants are given in Table V) are compared with the calculated values in Table IV.

Tabce VI The Calculated Values of Potential Curve.

R (10^{-8} cm)	ΔE (10^{-14} ergs) from (v)	from (vi)	from (viii)	from Table IV
3.70	7.08	4.47	6.70	6.20
2.64	242	253	298	328
2.12	2520	3690	2060	2090

From Table VI, we see the analytical form Eq. (37) fits best and Eq. (36) with constants $B=6.60\times10^{-8}$ ergs Å^n and $n=10.5$ fits next best. The agreement between ρ from our calculation and the one determined from experimental values by Buckingham is surprisingly good. The constant b differs by about three times from the b of Buckingham but this does not affect the form of potential curve appreciably.

The author is indebted to the suggestion of the problem and the valuable advice given by Assistant Professor Shu Ono.

Appendix A

We transform $F(R)$ the function of only the distance R from a point O, to the function of coordinates r and θ relative to another point O' separated from O by distance a (See Fig. A1). For this purpose, we expand $F(R)$ in the series of Legendre polynomials

$$F(R)=\rho_0(r)+\rho_1(r)P_1(\cos\theta)+\rho_2(r)P_2(\cos\theta)+\cdots \tag{A1}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of degree n. $P_n(\cos\theta)$'s satisfy the relation

$$\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{nm}. \tag{A2}$$

Multiplyting $P_n(\cos\theta)$ $\sin\theta$ $d\theta$ on both sides of Eq. (A1) and integrating over θ we get

$$\int F(R)P_n(\cos\theta)\sin\theta\, d\theta=\rho_n(r)\frac{2}{2n+1}. \tag{A3}$$

For $n=0$, 1, 2, 3 and 4, it becomes

$$\rho_0(r)=\frac{1}{2}\int F(R)\sin\theta\, d\theta \tag{A4}$$

$$\rho_1(r)=\frac{3}{2}\int F(R)P_1\sin\theta\, d\theta \tag{A5}$$

$$\rho_2(r)=\frac{5}{2}\int F(R)P_2\sin\theta\, d\theta \tag{A6}$$

$$\rho_3(r)=\frac{7}{2}\int F(R)P_3\sin\theta\, d\theta \tag{A7}$$

$$\rho_4(r)=\frac{9}{2}\int F(R)P_4\sin\theta\, d\theta \tag{A8}$$

where

$$P_0(\mu)=1,\ P_1(\mu)=\mu,\ P_2(\mu)=\frac{1}{2}(3\mu^2-1),$$

$$P_3(\mu)=\frac{1}{2}(5\mu^3-3\mu),\ \text{and}$$

$$P_4(\mu)=\frac{1}{8}(35\mu^4-30\mu^2+3).$$

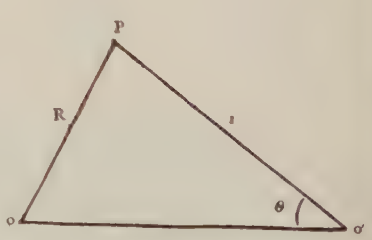

Fig. A1

From Fig. A1 we have

$$\cos\theta=\frac{r^2+a^2-R^2}{2ar} \tag{A9}$$

and

$$\sin\theta\, d\theta = \frac{1}{ar} R dR. \tag{A10}$$

Eq. (A4) is given by

$$\rho_0(r) = \frac{1}{2}\int_0^\pi F(R)\sin\theta\, d\theta = \frac{1}{2ar}\int_{|a-r|}^{|a+r|} F(R) R dR. \tag{A11}$$

Similarly, Eqs. (A5)–(A8) are expressed by

$$\rho_1(r) = \frac{3}{4}\frac{a^2+r^2}{a^2r^2}\int F(R) R dR - \frac{3}{4}\frac{1}{a^2r^2}\int F(R) R^3 dR \tag{A12}$$

$$\rho_2(r) = \frac{5}{2}\frac{a^2+r^2}{ar}\rho_1(r) - \frac{5}{2}\left(\frac{3}{4}\frac{(a^2+r^2)^2}{a^2r^2}+1\right)\rho_0 + \frac{15}{16}\frac{1}{a^3r^3}\int F(R) R^5 dR \tag{A13}$$

$$\rho_3(r) = \frac{7}{2}\left(\frac{5}{4}\frac{(a^2+r^2)^2}{a^2r^2}-1\right)\rho_1 - \frac{38}{85}\frac{(a^2+r^2)^3}{a^3r^3}\rho_0 + \frac{105}{32}\frac{(a^2+r^2)}{a^4r^4}\int F(R) R^5 dR$$

$$-\frac{105}{32}\frac{1}{a^4r^4}\int F(R) R^7 dR \tag{A14}$$

$$\rho_4(r) = -\frac{9}{2}\rho_2 + \frac{105}{16}\frac{(a^2+r^2)^3}{a^3r^3}\rho_1 - \frac{63}{8}\left(\frac{15}{16}\frac{(a^2+r^2)^4}{a^4r^4}+1\right)\rho_0$$

$$+\frac{945}{128}\frac{(a^2+r^2)^2}{a^5r^5}\int F(R) R^5 dR - \frac{315}{64}\frac{(a_6+r^2)}{a^5r^5}\int F(R) R^7 dR$$

$$+\frac{315}{256}\frac{1}{a^5r^5}\int F(R) R^9 dR. \tag{A15}$$

Appendix B

The electronic wave functions of argon atom is given by

$$\psi^a_{ns} = \frac{1}{\sqrt{4\pi}}\frac{f_{ns}(R)}{R} \tag{B1}$$

$$\psi^a_{np_0} = \sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R}\cos\theta \tag{B2}$$

$$\psi^a_{np_{\sin}} = \sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R}\sin\theta\cos\Phi \tag{B3}$$

$$\psi^a_{np_{\cos}} = \sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R}\sin\theta\sin\Phi \tag{B4}$$

where θ is equal to $\angle POO'$ in the Fig. B1, Φ is the angle between a certain fixed plane containing the line OO' and the plane POO'. Eqs. (B1)–(B4) are

the functions of coordinates R, Θ and Φ. We will express them as functions of coordinates r, θ and φ relative to O'.

I. Eq. (B1) for ns-state.

We expand $f_{ns}(R)/R$ as

$$f_{ns}(R)/R=\sum\rho_l^{ns}(r)P_l(\cos\theta)$$

and we get

$$\psi_{ns}^{a}=\frac{1}{\sqrt{4\pi}}\sum\rho_l^{ns}(r)P_l(\cos\theta) \tag{B5}$$

II. Eq. (B2) for np-state. From Fig. B1 we have Eq. (A8) and

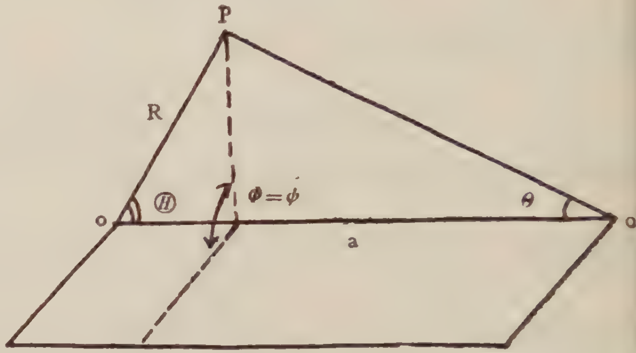

Fig. B1

$$\cos\Theta=\frac{a-r\cos\theta}{R} \tag{B6}$$

$$\sin\Theta=\frac{r}{R}\sin\theta. \tag{B7}$$

Inserting Eq. (B6) in (B2),

$$\psi_{np_0}^{a}=\sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R^2}(a-r\cos\theta).$$

Now putting

$$f_{np}(R)/R^2=\sum\rho_l^{np}(r)P_l(\cos\theta)$$

we get

$$\psi_{np_0}^{a}=\sqrt{\frac{3}{4\pi}}(a-r\cos\theta)\sum\rho_l^{np}(r)P_l(\cos\theta).$$

Using the formula for Legendre polynomials,

$$\cos\theta\,P_l(\cos\theta)=\frac{l+1}{2l+1}P_{l+1}+\frac{l}{2l+1}P_{l-1}$$

finally we get

$$\psi_{np_0}^{a}=\sqrt{\frac{3}{4\pi}}\sum\left(-\frac{l}{2l-1}r\rho_{l-1}^{np}(r)+a\rho_l^{np}(r)-\frac{l+1}{2l+3}r\rho_{l+1}^{np}(r)\right)P_l(\cos\theta). \tag{B8}$$

III. Eq. (B3) and (B4), $np_{\sin}$-state and $np_{\cos}$-state. Putting Eqs. (B3) and (B4) together and designating them as $\psi_{np_1}^{a}$,

$$\psi_{np_1}^{a}=\sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R}\sin\Theta\,{}^{\sin}_{\cos}\,\Phi$$

and inserting Eq. (B7) in it, we get

$$\psi_{np_1}^{a}=\sqrt{\frac{3}{4\pi}}\frac{f_{np}(R)}{R^2}r\sin\theta\,\frac{\sin}{\cos}\varphi$$

since $\Phi=\varphi$. And we put

$$f_{np}(R)/R^2=\sum\rho_l^{np}(r)P_l(\cos\theta).$$

Using the formula for the associated Legendre function

$$\sin\theta\,P_l(\cos\theta)=\frac{1}{2l+1}(P_{l+1}^1-P_{l-1}^1)$$

we get

$$\psi_{np_1}^{a}=\sqrt{\frac{3}{4\pi}}\frac{\sin}{\cos}\varphi\sum\left(\frac{1}{2l-1}\rho_{l-1}^{np}(r)-\frac{1}{2l+1}\rho_{l+1}^{np}(r)\right)P_l^1(\cos\theta). \tag{B9}$$

Accordingly the transformation is given by Eqs. (B5), (B8) and (B9). The ρ_{nl}'s in these equations are given by Appendix A.

Appendix C.

In Eq. (14), there are many integrals whose integrands are the products of a function of R, θ and Φ and a function of r, θ and φ. As in Appendix B the function of (R, θ, Φ) is transformed into the series of the function of (r, θ, φ). We shall deduce the forms of integrals in this Appendix. The transformed functions are expressed by Eqs. (B5), (B8) and (B9).

I. For the product of s-state and s-state.

As

$$\psi_{ns}^{b}=\frac{1}{\sqrt{4\pi}}\frac{f_{ns}(r)}{r}$$

the product may be written as

$$\int\psi_{ns}^{a}\psi_{ns}^{b}d\tau=\frac{1}{4\pi}\iint\rho_0(r)f_{ns}(r)rdrd\Omega,$$

where

$$d\Omega=\sin\theta\,d\theta d\varphi,$$

and we have

$$\int\psi_{ns}^{a}\psi_{ns}^{b}d\tau=\int\rho_0^{ns}(r)f_{ns}(r)rdr. \tag{C1}$$

II. For the product of s-state and p_0-state

As

$$\psi_{np_0}^{b}=\sqrt{\frac{3}{4\pi}}\frac{f_{np_0}(r)}{r}\cos\theta,$$

we get

$$\int \psi^a_{ns}\psi^b_{np_0}d\tau = \frac{\sqrt{3}}{4\pi}\iint \rho_1^{ns}(r)f_{np_0}(r)rdr\{P_1(\cos\theta)\}^2 d\Omega$$

$$= \frac{1}{\sqrt{3}}\int \rho_1^{ns}(r)f_{np_0}(r)rdr \tag{C2}$$

where we use the formula

$$\int \{P_n(\cos\theta)\}^2 d\Omega = \frac{4\pi}{2n+1}$$

III. For the product of p_0-state and s-state

As

$$\psi^b_{ns} = \frac{1}{\sqrt{4\pi}}\frac{f_{ns}(r)}{r}$$

we get

$$\int \psi^a_{np_0}\psi^b_{ns}d\tau = \frac{\sqrt{3}}{4\pi}\iint\left(a\rho_0^{np}(r) - \frac{1}{3}r\rho_1^{np}(r)\right)f_{ns}(r)rdrd\Omega$$

$$= \sqrt{3}\,a\int\left(\rho_0^{np}(r) - \frac{1}{3}\frac{r}{a}\rho_1^{np}(r)\right)f_{ns}(r)rdr. \tag{C3}$$

IV. For the product of p_0-state and p_0-state

As

$$\psi^b_{np_0} = \sqrt{\frac{3}{4\pi}}\frac{f_{np}(r)}{r}\cos\theta,$$

we get

$$\int \psi^a_{np_0}\psi^b_{np_0}d\tau = \frac{3}{4\pi}\iint\left(-r\rho_0^{np}(r) + a\rho_1^{np}(r) - \frac{2}{5}r\rho_2^{np}(r)\right)f_{np}(r)rdr\{P_1(\cos\theta)\}^2 d\Omega$$

$$= a\int\left(\rho_1^{np} - \frac{2r}{5a}\rho_2^{np}(r) - \frac{r}{a}\rho_0^{np}(r)\right)f_{np}(r)rdr. \tag{C4}$$

V. For the product of $np_{\sin}$-state and $np_{\sin}$-state, and for the product of $np_{\cos}$-state and $np_{\cos}$-state.

As

$$\psi^b_{np_1} = \sqrt{\frac{3}{4\pi}}\frac{f_{np}(r)}{r}\sin\theta\begin{matrix}\sin\\\cos\end{matrix}\varphi$$

we get

$$\int \psi^a_{np_1}\psi^b_{np_1}d\tau = \frac{3}{4\pi}\iint\left(\rho_0^{np}(r) - \frac{1}{5}\rho_2^{np}(r)\right)f_{np}(r)rdr\{P_1^1(\cos\theta)\}^2\left(\begin{matrix}\sin\\\cos\end{matrix}\varphi\right)^2 d\Omega$$

$$=\int\left(\rho_0^{np}(r)-\frac{1}{5}\rho_2^{np}(r)\right)f_{np}(r)rdr \tag{C5}$$

where the formula

$$\int\{P_n^m(\cos\theta)\cos m\varphi\}^2 d\Omega=\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}$$

is used.

For the products of ns-state and np_1-state, np_0-state and np_1-state, $np_{\sin}$-state and $np_{\cos}$-state etc. the integrals vanish from the integration over φ.

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On the East-West Effect of the Cosmic-Radiation in the Upper Atmosphere

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The fact that in the upper atmosphere there is a symmetry of the east west intensity of the cosmic-ray,[1] brings a difficulty to the Proton Primary Hypothesis which is assumed from the experimental results of the east-west asymmetry on each latitude and at each altitude from sea-level to 10,000m height.[2] To remedy the situation, Arley[3] assumed that there exist negative protons in the primaries. But the mechanism, by which negative proton directly gives the soft component, has no clearness. In order to make clear to what extent we can solve the problem of east-west effect from the standpoint which is now accepted—that is, primary proton gives π-meson and π meson decays into μ-meson which gives the soft component by disintegration,—we have investigated the angular distribution between the primary proton and the produced soft component.

I. Analysis of the Experiments

1. Some Assumptions.

In order to get the angular distribution adequate to explain the experimental results in the upper atmosphere ($\lesssim$ 1m H_2O), we analyse them under the assumptions as below;

i). Primary proton, colliding with the air nucleus, emits π-meson and then after π-μ-decay, μ-meson gives the soft component. The probability of these processes is proportional to the primary proton intensity and depends upon the magnetic cut-off energy of proton ε_c. It has the form,

$$\pi\, d\Omega\, dx = b\left[\varepsilon_c(\theta_0)\right] \exp\left(-x/l\cos\theta_0\right) P(\theta_0)\, dx\, d\Omega$$

$$l = 1.25m \ \mathrm{H_2O}. \tag{1}$$

$P(\theta_0)$ is the incident flux of primary proton, the direction of which is θ_0 with the vertical.

ii). The intensity of soft components produced by the μ-mesons varies cascade-likely. As the soft intensity at the distance t from the point where nuclear events happen, we set approximately,

$$f(t) = a_1 t \exp\left(-a_2 t\right). \tag{2}$$

The validity of this approximation may be allowed phenomenologically, since a_1 and a_2 can be determined to fit the experiments. At $t=0$, $f(t)$ becomes zero. This corresponds to the fact that the nuclear collision of proton does not give the soft component directly, but π-μ-decay and $\mu-e$ decay precede.

iii). For the angular distribution function $R(\theta)$, θ being the angle between the primary proton and the produced soft component, we assume,

$$d\Omega R(\theta)=d\Omega\times\begin{cases}\text{const.} & (|\theta|\leq\omega)\\ 0 & (|\theta|>\omega)\end{cases}\tag{3}$$

2. Determination of the Angular Spread. Our first purpose is the estimation of the quantity ω from the analysis of the experiments.

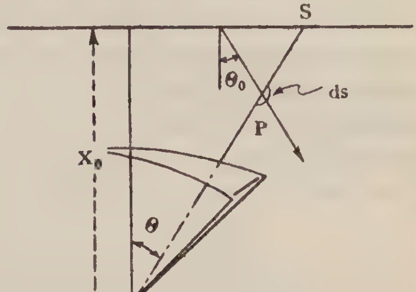

Fig. 1.

In Fig. I, the soft component in the direction θ found at the point T, has its origin on the line TS. The probability of the soft component being produced in the infinitesimal length ds about P on TS, is

$$\propto ds\int be^{-x/l\cos\theta_0}R(\theta_0-\theta)d\theta_0 P(\theta_0)=\frac{dx}{\cos\theta}\int bc^{\,x/l\cos\theta_0}P(\theta_0)R(\theta-\theta_0)d\theta_0.\tag{4}$$

If the produced soft component multiplicates by the formula (2) along its path to the point T, the soft intensity in the direction θ at T, is

$$S(x_0,\theta)\propto\iint e^{-x/l\cos\theta_0}R(\theta_0-\theta)d\theta_0 f\left(\frac{x_0-x}{\cos\theta}\right)P(\theta_0)dx/\cos\theta$$

$$=\frac{a_1}{\cos^2\theta}\int_0^{x_0}dx\int be^{\,x/l\cos\theta_0}R(\theta_0-\theta)(x_0-x)\,e^{-a_2(x_0-x)/\cos\theta}d\theta_0P(\theta_0).\tag{5}$$

The integration of (5) with respect to x gives;

$$S(x_0,\theta)\propto\frac{a^1}{\cos^2\theta}\int R(\theta_0-\theta)\frac{b}{\beta}(x_0+e^{-\beta x_0}/\beta-1/\beta)e^{-a_2x_0/\cos\theta}d\theta_0P(\theta_0)\tag{6}$$

$$\beta=(1/l\cos\theta_0-a_2/\cos\theta).$$

In the above discussion, our treatment is two-dimensional one, while essentially it must be three-dimensional. The effect of the three-dimensional treatment, however, say, the effect perpendicular to this sheet of this paper, is almost same as for each zenithal angle θ. So, it does not bring any appreciable correction when we discuss the relative magnitude of the east-west intensity. But the spread of the angular distribution (ω in (3)) becomes smaller by 8% than in the case where the treatment is three-dimensional.

Now we determine the value of parameters by the comparison with the experiments. As the soft intensity, we take the value of its vertical intensity at 45°N.[4] In (6), we take $\theta=0$, and then replace the factor $P(\theta_0)b$ by its mean value on θ_0. Putting it outside the integral, we determine the values of a_1 and a_2 to fit the experimental curve. b is, from (1), some quantity which depends upon the multiplicity and the energy of the π-meson production.

As the experiments to be compared with (6), we take that of Johnson et al.[5] For $P(\theta_0)$, in correspondence with their experiment, we take the mean value on

South-East-North, North-West-South directions at each zenithal angle θ_0 from the graph obtained by Vallarta.[6] $P(\theta_0)b$ in the integrand of (6) is the quantity which is responsible for the non-symmetry of the east-west intensity. By taking a definite θ_0, we can obtain the corresponding magnetic cut-off energy and then total flux of proton, $P(\theta_0)$, but b cannot be determined in this way. So, we assume that the relative magnitude of $P(\theta_0)b$ is proportional to the vertical intensity at each latitude, in which the magnetic cut-off energy as to the vertical incident is same as the definite θ_0 at the latitude of the Johnson's experiments. From the experiment of Neher et al.[7], it becomes,

$$P(\theta_0)b \propto \{\varepsilon_c(\theta_0)\}^{-0.75}. \tag{7}$$

The east-west effect T is expressed as following;

$$T=2\frac{S_W(\theta, x_0)-S_E(\theta, x_0)}{S_W(\theta, x_0)+S_E(\theta, x_0)}. \qquad (\lambda=20°N) \tag{8}$$

In the experiments of Johnson et al., the highest observing altitude is 0.33m H_2O, and they have taken the mean value of the results at the highest and the lower altitude. We select the results at $x_0=0.5$m H_2O. We can assume that the variation of the altitude in the neighbourhood of this height does not affect the east-west effect appreciably. Thus, we determine the angular distribution fuction $R(\theta)$ involved in S in agreement with the experiments of Johnson et al., $T\sim 10\%$. Then the value of ω is,

$$\omega \simeq 0.87 \quad \text{radian}. \tag{9}$$

Our calculation is made only upon the soft component, while the result of Johnson et al. is that of the total intensity. The hard components and the surviving protons almost conserve their direction, while the value given in (9) is valid only about the soft component. And so, the value in (9) may be taken as the minimum of the angular extension. But, in order to enter the region of these problems, our methods of analysis is too insufficient. More accurate treatments, in which the energy distribution of meson production and other effects are considered, are required. The above estimation may be correct only in the region $\lesssim$2m H_2O.

II. Theoretical Treatments.

In correspondence with the treatments of the above section, we estimate the angular distribution between the primary proton incident vertically on the top of the atmosphere and the produced soft origin (electron), at the depth xm H_2O, semi-theoretically.

Here, the energy spectrum of the vertical proton at xmH_2O depth is assumed to be

$$d\varepsilon\, P(\varepsilon, x)=N\varepsilon^{-\gamma-1}d\varepsilon\, e^{-x/l}. \qquad \gamma=1.8 \tag{10}$$

Under this assumed spectrum, the contribution from the low energy region is overestimated.[9] But it does not affect appreciably if the magnetic cut-off energy is large.

i) π-meson production.

It is now assumed that the emission of π-meson in nucleon-nucleon collision is isotropic in the mass-system[8] and the angular distribution and the energy distribution is independent each other. Then, the probability of the π-meson being produced by the proton of energy ε in the energy range E_0, E_0+dE_0 and in the angular range θ_0, $\theta_0+d\theta_0$, is*

$$\Psi(\varepsilon, E_0, \theta_0)\,dE_0 d\Omega_0 = C\varepsilon^s E_0^r dE_0 (1-u^2)/(1-u\cos\theta_0)^2\, d\Omega_0/4\pi$$

$$u = p/\varepsilon + K \tag{11}$$

where p and K are the momentum and mass of proton respectively. u is the relative velocity of the mass-system to the laboratory system and we put $s=-r=1/3$. This values of the constants are consistent with the meson spectrum in the lower atmosphere. The probability of the production of π-meson at the depth x_0, in the direction θ with the vertical and in energy range E_0, E_0+dE_0 is,

$$dE_0 d\Omega\, dx_0 \pi(E_0, \theta_0, x_0) = NC\, e^{-x_0/l}\frac{dx_0}{a}\int_t^\infty d\varepsilon\, \varepsilon^{-\gamma-1+s}E_0^{r-1}(\varepsilon+K)\frac{d\Omega_0}{[\varepsilon(1-\cos\theta_0)+K]^2}\frac{K}{2\pi}$$

$$= NC\, e^{-x_0/l}\frac{K}{2\pi a}\,d\Omega\, dE_0 E_0^{r-1} dx_0 \int_t^\infty d\varepsilon\, \varepsilon^{-\gamma-1+s}(\varepsilon+K)\frac{1}{[\varepsilon(1-\cos\theta_0)+K]^2} \tag{12}$$

In (12), we replace $u=p/\varepsilon+K$ by $\varepsilon/\varepsilon+K$. This approximation may be allowed at the latitude of high magnetic cut-off ($E_c > 10$BeV), and at the high altitude. We put $t=E_c \exp(-x_0/l)$ as the lower limit of the integral, considering the energy loss in the air. This is not correct when the energy of the produced π-meson is $\gtrsim E_c$. But the μ-meson, produced by such energetic π-meson, has so high energy that it does not affect the soft component in the upper atmosphere. Under the above approximation, (12) can be integrated and the result is,

$$\pi(E_0, \theta_0, x_0)\,dE_0 dx_0 d\Omega = NC\, e^{-x_0/l} d\Omega_0 \frac{K}{2\pi a} dE_0 E_0^{r-1}\frac{2}{K^2}(I_1\cos\theta_0 + I_2)$$

$$I_1 = \left[\frac{1}{\sqrt{t}} + \frac{(1-\cos\theta_0)\sqrt{t}}{2[t(1-\cos\theta_0)+K]} + \frac{3}{4}\sqrt{\frac{1-\cos\theta_0}{K}}\left(2\tan^{-1}\sqrt{\frac{t(1-\cos\theta_0)}{K}} - \pi\right)\right]$$

$$I_2 = \left[\frac{K}{3t^{3/2}} - \frac{1-\cos\theta_0}{\sqrt{t}} + \frac{(1-\cos\theta_0)^{3/2}}{2\sqrt{K}}\left(\pi - 2\tan^{-1}\sqrt{\frac{t(1-\cos\theta_0)}{K}}\right)\right] \tag{13}$$

ii) π-μ-decay.

As the mass of the π-and μ-meson, we take $\varkappa=286$m and $\mu=217$m respec-

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* The form of the angular distribution $(1-u^2)/(1-u\cos\theta_0)^2\, d\Omega/4\pi$ is generally obtained if (m_π/E) is neglected compared with 1.

tively. In the rest-system of π-meson, the energy and the momentum of the μ-meson is denoted by ε_0 and $\boldsymbol{p}_0$ respectively and θ_0 is the angle between $\boldsymbol{p}_0$ and the direction of π-meson. The corresponding quantities in the laboratory system is denoted by ϵ, $\boldsymbol{p}$ and θ, respectively. The velocity of π-meson is $u'=p_0/E_0$. Then,

$$\epsilon=(\epsilon_0+u'p_0\cos\theta_0)/\sqrt{1-(u')^2} \qquad |\boldsymbol{p}_0|=p_0$$

From this, the energy distribution of the produced μ-meson is,

$$f(\epsilon)d\epsilon=\begin{cases} d\epsilon/aE_0 & E_0\geq\epsilon\geq(1-a)E_0. \\ 0 & (1-a)E_0>\epsilon. \end{cases} \tag{14}$$

In the previous cases, when one particle decays into two particles with negligible masses, the energy distribution has the form $d\epsilon/dE$ and so the mean energy becomes $0.5\,E_0$. While in the case of π-μ-decay, on account of the small mass difference of π-and μ-meson, the mean energy of μ-meson becomes larger. Under the approximation

$$\epsilon_0=225\text{m} \gg p_0=61\text{m},$$

the obtained angular distribution indicates that μ-meson cannot deviates from the direction of π-meson by the amount larger than $\theta_{\max}\sim(p_0/\mu)(\varkappa/p)$ (p is the momentum of π-meson). When we restrict ourselves to the case of $E_0\gtrsim2\varkappa$, $\theta_{\max}$ is $\lesssim0.1$ radian. Since the π-meson of $E_0\lesssim2\varkappa$ does not contribute energetically, we may assume that the directions are all conserved in the case of the π-μ-decay. As is clear from (14), the energy range of the produced μ-meson is so small that the energy spectrum of μ-meson produced by the π-meson of energy E_0, is approximated by,

$$\delta(a'E_0-\epsilon)d\epsilon \qquad a'=0.78. \tag{15}$$

Next, the intensity of μ-meson at x, which is produced at x_0 with the energy $\epsilon+\beta'(x-x_0)$ and with the angle θ with the vertical, decreases on account of the collision and the decay by a factor,

$$\left\{\frac{x_0}{x}\,(\epsilon/\epsilon+\beta'(x-x_0))\right\}^{b'/\epsilon+\beta'x}$$

$$b'=b/\cos\theta,\quad \beta'=\beta/\cos\theta \quad b=(\mu c/\tau_0)(x_0/\rho)=1.1\text{ BeV} \tag{16}$$

$$\beta=0.20\text{ BeV/m }H_2O.$$

So the intensity of μ-meson at x, in θ-direction and in the energy range ϵ, $\epsilon+d\epsilon$, is

$$d\Omega d\epsilon\,\mu(\epsilon,\theta,x)=NC\frac{K}{2\pi a}\frac{2}{K^2}\int e^{-x_0/l}dE_0E_0^{-\gamma-1}(I_1\cos\theta+I_2)$$

$$\times\int\delta(\epsilon+\beta'(x-x_0)-a'E_0)\left\{\frac{x_0}{x}\,\frac{\epsilon}{\epsilon+\beta'(x-x_0)}\right\}^{b'/\epsilon+\beta'x}d\Omega d\epsilon$$

$$=NC\frac{d\Omega d\epsilon}{\pi Kaa'}\int e^{-x_0/l}\,dx_0\left\{\frac{x_0}{x}\frac{\epsilon}{\epsilon+\beta'(x-x_0)}\right\}\left\{\frac{\epsilon+\beta'(x-x_0)}{a'}\right\}^{\prime\,-1}(I_1\cos\theta+I_2). \tag{17}$$

iii) μ-meson decay.

The natural decay of μ-meson is associated with one electron (or positron) and two neutral particles. We take the scalar interaction

$$H'=g(\psi_e^*\beta\psi_\nu)(\psi_\nu^*\beta\psi_\mu)+c.c., \tag{18}$$

where ψ_e, ψ_ν and ψ_μ is the wave function of electron, neutrino and μ-meson respectively. In the rest-system of μ-meson, the energy distribution of electron (or positron) is

$$f(E)dE\propto E^2(3\mu-4E)dE, \tag{19}$$

where the mass of neutral μ-meson $\mu_0=0$. And the directional distribution of the emitted electron is isotropic. In the decay process of the μ-meson of energy ε with the velocity $u(u=p^0/\mu)$, the energy distribution of electron (positron) which is in the energy range E, $E+dE$, and in the angle θ with the direction of μ-meson, becomes,

$$I(\theta,E)dEd\Omega=\left(\frac{2}{\mu}\right)^4\frac{1}{\sqrt{1-u^2}}(1-u\cos\theta)E^2\left(3\mu-4\frac{1-u\cos\theta}{\sqrt{1-u^2}}E\right)dEd\Omega/4\pi$$

$$E\leqq\frac{\mu}{2}\sqrt{1-u^2}/1-u\cos\theta. \tag{20}$$

After integration on energy, the angular distribution becomes,

$$d\Omega I(\theta)=(\sqrt{1-u^2}/1-u\cos\theta)^2d\Omega/4\pi. \tag{21}$$

Under the approximation $(\mu/\varepsilon)^2\ll 1$, the energy spectrum is obtained by integrating on θ,

$$dEI(\epsilon,E)=\left(\frac{5}{3}-3\frac{E^2}{\epsilon^2}+\frac{4}{3}\frac{E^3}{\epsilon^3}\right)\frac{dE}{\epsilon}. \tag{22}$$

It may be expected that the forms of the spectrum obtained above do not vary appreciably with the type of the interaction in (13)[10]. As is seen from (21), in the case of μ-meson decay, the electrons are scarcely emitted in the angle larger than

$$\theta_{\max}\simeq\mu/p. \tag{23}$$

For the simplification of our calculation, we accept the following approximations; taking the mean value of the angular deviation of the electron emitted with the angular distibution (23) on one azimuthal plane, we get $<\theta>_{AV}=\frac{1}{\sqrt{2}}\mu/p$. For the electron in θ-direction we take the mean contribution of the μ-meson intensities with the directions of $(\theta-<\theta>_{AV})$ and $(\theta+<\theta>_{AV})$

iv) Angular distribution.

Taking into account that the rate of μ-meson decay in the distance dx is $(b'/\varepsilon_\mu x)dx$, the intensity of electron with energy between E and $E+dE$ which is emitted in the direction θ at depth x, becomes,

$$dE\,dx\,d\Omega_\Theta/4\pi\cdot S(E,\theta,x)=\int_E<\mu(E,\theta,x)\frac{b'}{\epsilon x}>_{Av}d\epsilon\cdot dx\,dE\,d\Omega_\Theta/4\pi \qquad (24)$$

where the symbol $<\ >_{Av}$ indicates such meán values of the intensity as stated at the end of iii). The energy transferred to the electron is

$$dx\,d\Omega_\Theta\int E\,S(E,\theta,x)dE=dx\frac{d\Omega_\Theta}{4\pi}\frac{7}{20}\int_E<\mu(\epsilon,\theta,x)\frac{b'}{x}>_{Av}d\epsilon$$

$$=dx\,d\Omega_\Theta/4\pi\cdot E_s(\theta,x). \qquad (25)$$

The value ω estimated in the above section does not correspond to the angular distribution of the soft origin, but to that of the transferred energy to the soft. For the angular distribution is determined about the soft intensity which varies cascade-likely, while the cascade intensity is proportional to the energy of the soft origin. Using (25), the mean angle of the soft component to the vertical is,

$$<\theta_s>_p=\int\theta E_s(\theta,x)d\theta/\int E_s(\theta,x)d\theta. \qquad (26)$$

In this integral, integration is done with respect to $d\theta$, not to $d\Omega_\Theta$ in corresponding with the treatment of the above section. For the comparison, we put $\omega=0.87$ radian and then we define the quantities $<\omega>_p$ and $<\theta_\mu>_p$, respectively, as

$$<\omega>_p=\int^\omega\theta\cdot d\theta/\int^\omega d\theta \qquad (26')$$

$$<\theta_\mu>_p=\int\theta\mu(\theta,x)d\theta/\int\mu(\theta,x)d\theta. \qquad (26'')$$

And the results at $x=0.5$m H_2O are shown in Table I.

Table I.

$<\theta_s>_p$	$<\theta_\mu>_p$	$<\omega>_p$
0.197	0.155	0.435

III. Discussions.

So far we have neglected the contribution due to the neutral meson. In fact, the amount of the absolute intensity of the soft component in the upper atmosphere cannot be obtained, if the neutral mesons are not taken into account. But the neutral meson may be expected to be produced with the same distribution as the case of the charged meson in the nucleon-nucleon collision. So, even in the case where the effect of the neutral meson is added, the situation does not essentially change in our problems.

Now, $<\theta>$ shown in Table I, is the values obtained from the estimation only of the vertically incident protons. As for the non-vertically incident ones, the spread of this quantity is not expected to change appreciably. Now the angular spread gives only an amount of 45% of that required from the experiment. The rough estimation of the east-west effect of the soft component only, setting the values of ω equal to 0.39 radian in correspondence with the $<\theta_s>$ which is obtained from Table I gives

$$T_s(55°)=38\%.$$

Next, if we assume that the proton intensity decreases by $\exp(-x/l)$ in conserving the direction in its path in atmosphere, the east-west effect is estimated as

$$T_{proton} \simeq 80\%. \quad (x=0.5\mathrm{mH_2O})$$

Concering to the μ-meson intensity, an appreciable variation with the angle with the vertical may be expected. But, in the rough approximation——that is, taking $<\theta_\mu>$ in Table I as the angular spread of the μ-meson intensity, assuming the μ-meson intensity to be proportional to the flux of the proton multiplied by a-third power of the magnetic cut-off energy of the proton due to the multiplicity of the π-meson production and replacing the variation of the μ-meson intensity with the atmospheric depth by the experimental curve transformed by the factor $(x/\cos\theta)$——we get,

$$T_\mu \simeq 50\% \qquad (x=0.5\mathrm{mH_2O}).$$

If these three components are mixed by the ratio $a:b:c$, the average value of the east-west effect is

$$<T>=\frac{aT_s+bT_\mu+cT_p}{a+b+c}.$$

For example, in the case of $a:b:c$ $1:1:\frac{1}{2}$, it becomes,

$$<T>_{ob} \simeq 50\%.$$

Several effects may be considered in order to save the difficulties in the problem of the east-west symmetry of the cosmic ray intensity in the upper atmosphere. i). If there exist negative protons in the primary, the intensity of negative proton must be about 40% of that of the primary in order to explain the results of Johnson et al., while this is in contradiction with the experiment about the east-west effect of the hard component in the lower atmosphere ($\lesssim$ 10,000m). ii). Deflection of the charged particles due to the geomagnetic effect is negligible.[11] iii). Star is a process in which particles do not conserve the direction, but the energy of the particles produced in it being about 10^7ev, they do not penetrate the counter wall. So the star phenomenon may not affect the experiment. iv). Heavy nuclei involved in the primary radiation give a contribution which diminishes the east-west effect, but to explain the phenomenon,

the now accepted amount of heavy nuclei, $H:He=4:1$[12] is not sufficient.

The discussions above stated is based upon rather rough estimation. But we may conclude that the above mentioned processes are not essential to explain the east-west effect in the upper atmosphere. So, we may expect that special processes of the meson production in the nucleon-nucleon collision play a role, say, the plural production, besides the multiple one or the large backward distribution of the produced meson in proton-neutron collision.

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On the Covariant Formalism of the Quantum Theory of Fields. I.[1]

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As a result of the new formalism of Tomonaga[2] and Schwinger[3] the quantum theory of fields has made very notable advances. Especially it has given many beautiful results when applied to the system of electrons interacting with radiation field.[4] In this case we have found no ambiguities in formal aspects. On the other hand in the case of mesons interacting with radiation field,[5] it appears that there remain some ambiguities concerning with the conditions of integrability and the Lorentz invariance.

In the theory of Tomonaga and Schwinger the generalized Schroedinger equation is given by

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma_P}=H(P)\Psi[\sigma]$$

where $H(P)$ is a density of interaction energy at a world point P. In case of electron-radiation field, $H(P)$ is relativistically invariant and at the same time it satisfies the condition of integrability. On the other hand in case of meson-radiation field, $H(P)$ is not relativistically invariant and also it does not satisfy the integrability condition with regard to $\delta\Psi/\delta\sigma$. To remove these formal difficulties, Tomonaga and Kanesawa have added one term which depends on the normal of the surface σ at the point considered, so that the modified H satisfies the both conditions of integrability and Lorentz invariance.

Therefore it is worth while to try to make clear these circumstances. It is the first purpose of this paper. In the second place we shall consider the following problem: if any system of fields is invariant for some transformation groups, what happens in the interaction representation concerning with the character of invariance mentioned above?

§ I. Classical Theory (Lagrange Formalism.)

Let us consider any one Lorentz system in the flat space-time, and denote the coordinate value of a world point P with regard to this coordinate system as follows:

$$x^k=(x,y,z,t)^{6)}\ k=1,2,3,0.$$

In general Latin suffix refers to Lorentz system. Furthermore we introduce a system of curvilinear coordinate (ξ-system) in the same space-time, and denote the coordinate value of P referred to this ξ-system as follows :[7]

$$\xi^{\mu}=(\xi^{\dot{1}}\xi^{\dot{2}}\xi^{\dot{3}}\xi^{\dot{0}}).$$

The Greek suffix refers to ξ-system. The dotted figure (for example $\dot{1}$, $\dot{2}$,……) means that this figure refers to ξ-system. And the dashed figure (for example $\bar{\mu}$, $\bar{k}$) means that this figure expresses 1, 2, or 3. Further, we shall use the summation convention of the usual tensor calculus.

Denoting the fundamental metric tensor in the Lorentz system by

$$g^{\bar{k}\bar{l}}=g_{\bar{k}\bar{l}}=\delta_{kl},\quad g^{\bar{k}0}=g_{\bar{k}0}=0\quad g^{00}=g_{00}=-1,$$

we have

$$ds^2=g_{kl}dx^k dx^l.$$

Putting

$$\frac{\partial x^k}{\partial \xi^{\mu}}=h^k_{.\mu}\quad \frac{\partial \xi^{\mu}}{\partial x^k}=h^{\mu}_{.k}\quad \gamma_{\mu\nu}=g_{kl}h^k_{.\mu}h^l_{.\nu}\quad \gamma^{\mu\nu}=g^{kl}h^{\mu}_{.k}h^{\nu}_{.l}$$

we have

$$ds^2=\gamma_{\mu\nu}d\xi^{\mu}d\xi^{\nu}$$

and

$$h^k_{\mu}h^{\mu}_l=\delta^k_l\quad h^{\mu}_k h^k_{\nu}=\delta^{\mu}_{\nu}.$$

Raising and lowering suffices of h is made in the usual way of tensor calculus. Further one easily sees it to be unnecessary to distinguish between $h^k_{.\mu}$ and $h_{\mu}^{\ k}$ etc.

Using the above notation, the functional determinant[8]

$$D=\frac{\partial(x^1\cdots x^0)}{\partial(\xi^{\dot{1}}\cdots\xi^{\dot{0}})}=\frac{\partial(x)}{\partial(\xi)}$$

can be expressed as

$$D=det(h^k_{\mu})=\sqrt{-\gamma}$$

where

$$\gamma=det(\gamma_{\mu\nu}).$$

Now let us denote the field quantities defined in the Lorentz system (calling x-system for simplicity) as $Q_A(x)\,(A=1,2,\cdots N)$. These may be spinors or tensors. The suffix A refers to x-system.

Consider a physical system described by a Lagrange function

$$L=L(Q_{A,k}Q_A),\quad Q_{A,k}=\frac{\partial Q_A(x)}{\partial x^k}.$$

Then the action integral is given by

$$I=\int L(dx)^4=\int DL\{Q_A,(h^\mu_k Q_{A,\mu})\}(d\xi)^4$$

$$Q_{A,\mu}=\frac{\partial Q_A(x(\xi))}{\partial \xi^\mu} \qquad (dx)^4=dx^0dx^1dx^2dx^3$$

$$(d\xi)^4=d\xi^0d\xi^1d\xi^2d\xi^3.$$

Putting

$$\mathfrak{L}=\mathfrak{L}\{Q_A,\ Q_{A,\mu}h^k_\mu\}=D\cdot L$$

we have

$$I=\int\mathfrak{L}(d\xi)^4=\int L(dx)^4.$$

The field equations are given by

$$\frac{\delta I}{\delta Q_A(\xi)}\equiv[\mathfrak{L}]_{Q_A}\equiv\frac{\partial\mathfrak{L}}{\partial Q_A}-\frac{\partial}{\partial\xi^\mu}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\mu}}\right)=0. \tag{1}$$

If we define

$$\mathfrak{P}^{A,\mu}=\frac{\partial\mathfrak{L}}{\partial Q_{A,\mu}}=DP^{A,\mu} \qquad P^{A,\mu}=\frac{\partial L}{\partial Q_{A,\mu}}=h^\mu_k\frac{\partial L}{\partial Q_{A,k}}$$

then (1) can be written

$$\frac{\partial\mathfrak{P}^{A,\mu}}{\partial\xi^\mu}-\frac{\partial\mathfrak{L}}{\partial Q_A}=0.$$

Introducing the Christoffel's 3-index symbols

$$\Gamma^\lambda_{\mu\nu}=h^\lambda_l\frac{\partial h^l_\nu}{\partial\xi^\mu}=h^\lambda_l\frac{\partial h^l_\mu}{\partial\xi^\nu}$$

(1) can be written as

$$\frac{\partial P^{A,\mu}}{\partial\xi^\mu}+\Gamma^\mu_{\mu\nu}P^{A,\nu}-\frac{\partial L}{\partial Q_A}=0. \tag{1'}$$

This form is invariant for the transformation of ξ-system (we shall call this ξ-transformation for simplicity), noticing that $P^{A,\mu}$ is a contravariant vector for this transformation. Furthermore we can easily see that (1′) is Lorentz covariant.

Next we shall consider the relationship between the field equations in the x-system and (1).

This is

$$[\mathfrak{L}]_{Q_A}=D[L]_{Q_A}\equiv D\left\{\frac{\partial L}{\partial Q_A}-\frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial Q_{A,k}}\right)\right\}.$$

Therefore $[\mathfrak{L}]_{Q_A}=0$ trivially implies $[L]_{Q_A}=0$, and vice versa.

Now we restrict the Lagrangian L to be of the form

$$L=\frac{1}{2}C^{AB,kl}Q_{A,k}Q_{B,l}+\frac{1}{2}\{C^{A,k}(Q)Q_{A,k}+Q_{A,k}C^{A,k}(Q)\}+C(Q) \tag{2}$$

where $C^{AB,kl}=C^{BA,lk}$ are constants.

In this case, (1′) becomes

$$C^{A,B\mu\nu}(\xi)Q_{B,\mu\nu}+(\cdots\cdots)=0 \quad C^{AB,\mu\nu}(\xi)=C^{AB,kl}h_k^{\mu}h_l^{\nu}$$

where $+(\cdots\cdots\cdots)$ expresses some function of Q_A, $Q_{B,\mu}$, and ξ^{μ}, but not of the second derivatives of Q. If we put $\varphi(\xi)=0$ as a characteristic of the above partial differential equations, then the equation to be satisfied by φ is

$$F\equiv det(S^{AB})=0, \quad S^{AB}=C^{AB,\mu\nu}(\xi)\varphi_{,\mu}\varphi_{,\nu} \tag{3}$$

according to the theory of partial differential equation.

Now it is easily seen that F must be invariant for ξ-transformation. Therefore F depends only on the invariant combination $\gamma^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$.

Further, from the very definition of F, it must be a homogeneous function of $2N$-th degree with regard to $\varphi_{,\mu}$. Therefore

$$F=\text{const.}\times(\gamma^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})^{N}=0.^{9)}$$

That is, the equation of the characteristic runs as follows

$$\gamma^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}=g^{kl}\varphi_{,k}\varphi_{,l}=0. \tag{4}$$

This means that the characteristic of the field equation is a usual light cone, as the consequence of the assumption (2).

§ 2. Transformation Group and Conservation Law.[10]

We shall now consider two types of transformation groups.

i) Consider the following infinitesimal transformation.

$$x^k\to x'^k=x^k+\delta x^k \tag{5.1}$$

$$Q_A(x)\to Q'_A(x')=Q_A(x)+\delta Q_A(x) \tag{5.2}$$

while ξ remains unaltered. In (5) x and x' represent the coordinate values of the same point referring to the x-systems before and after the transformation (5) respectively. In general we shall define two types of variation of any function $\Phi(x)$ as

$$\delta\Phi(x)=\Phi'(x')-\Phi(x)$$

$$\delta^*\Phi(x)=\Phi'(x)-\Phi(x).$$

In the former equation, x' and x correspond to the same world point, whereas, in the latter, the point x of $\Phi(x)$, and that of $\Phi'(x)$ differ in the following manner; the new coordinate value of the former point is equal to the old coordinate value of

the latter point. Therefore in the latter equation $\Phi'(x)$ and $\Phi(x)$ refer to two different points respectively. The relation of $\delta\Phi$ and $\delta^*\Phi$ is easily obtained from the definition

$$\delta\Phi=\delta^*\Phi+\frac{d\Phi}{dx^k}\delta x^k$$

and further we have the following formula

$$\delta^*\frac{d\Phi}{dx^k}=\frac{d}{dx^k}\delta^*\Phi.$$

Now we shall assume the action integral I to be invariant for the transformation (5). In this case; if we take x as independent parameters, then ξ becomes unnecessary and we have the following identity:

$$\delta I=\int_\Omega\left((\delta L+L\frac{d\delta x^k}{dx^k}\right)(dx)^4=\int_\Omega\left\{\delta^*L+\frac{d}{dx^k}(L\delta x^k)\right\}(dx)^4\equiv 0.$$

By virtue of arbitrariness of the domain Ω, it holds

$$\delta^*L+\frac{d}{dx^k}(Ldx^k)\equiv 0. \tag{6}$$

where

$$\delta^*L=\frac{\partial L}{\partial Q_A}\delta^*Q_A+\frac{\partial L}{\partial Q_{A,k}}\delta^*Q_{A,k}.$$

(6) can be rewritten

$$[L]_{Q_A}\delta^*Q_A+\frac{\partial}{\partial x^k}\{P^{Ak}\delta Q_A-T^k_l\delta x^l\}\equiv 0, \tag{7}$$

where

$$P^{Ak}=\frac{\partial L}{\partial Q^{A,k}}\quad T^k_l=P^{A,k}Q_{A,l}-\delta^k_l L. \tag{8}$$

Multiplying the both sides of (7) with D and using the identity $\frac{\partial}{\partial\xi^\mu}(Dh^\mu_k)\equiv 0$, we obtain

$$[\mathfrak{L}]_{Q_A}\delta^*Q_A+\frac{\partial}{\partial\xi^\mu}\{\mathfrak{P}^{A,\mu}\delta Q_A-\mathfrak{T}^\mu_l\delta x^l\}\equiv 0. \tag{7'}$$

where

$$\mathfrak{T}^\mu_l=Dh^\mu_k T^k_l. \tag{9}$$

On the other hand if we take ξ as independent parameters, then

$$\delta x^k=x'^k(\xi)-x^k(\xi)=\delta^*x^k$$

$$\delta Q_A=Q'_A\{x'(\xi)\}-Q_A\{x(\xi)\}=Q'_A(\xi)-Q_A(\xi)=\delta^*Q_A$$

where

$$\delta^* Q_A = \delta Q_A - \frac{\partial Q_A}{\partial \xi^\mu} \delta\xi^\mu$$

(in this case $\delta\xi^\mu = 0$).

In the same way we have

$$\delta h^k_\mu = \frac{\partial}{\partial \xi^\mu} \delta^* x^k = \frac{\partial \delta x^k}{\partial x^l} h^l_\mu .$$

(This represents that h^k_μ is a contravariant vector for the transformation (5)). Then, from

$$\delta I = \int_\Omega \delta \mathfrak{L} (d\xi)^4 = 0.$$

we obtain $\delta\mathfrak{L} \equiv 0$ instead of (6), that is

$$\delta\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial Q_A} \delta Q_A + \frac{\partial \mathfrak{L}}{\partial Q_{A,\mu}} \delta Q_{A,\mu} + \frac{\partial \mathfrak{L}}{\partial h^k_\mu} \delta h^k_\mu \equiv 0.$$

This can be transformed into the form

$$[\mathfrak{L}]_{Q_A} \delta Q - \frac{\partial}{\partial \xi^\mu}\left(\frac{\partial \mathfrak{L}}{\partial h^k_\mu}\right) \cdot \delta x^k + \tag{10}$$

$$+ \frac{\partial}{\partial \xi^\mu}\left(\mathfrak{P}^{A\mu} \delta Q_A + \frac{\partial \mathfrak{L}}{\partial h^k_\mu} \delta x^k\right) \equiv 0.$$

ii) Next we consider the second type of infinitesimal transformation.

$$\xi^\mu \to \xi^{\mu\prime} = \xi^\mu + \delta\xi^\mu \tag{11}$$

while x^k remains unaltered. Because of the definition of Q_A, we have $\delta Q_A = 0$. Therefore if we take ξ as independent parameters, then we have

$$\delta^* Q^A = -Q_{A,\mu} \delta\xi^\mu, \qquad \delta^* x^k = -h^k_\mu \delta\xi^\mu$$

$$\delta h^k_\mu = -\frac{\partial \delta\xi^\nu}{\partial \xi^\mu} h^k_\nu \qquad \delta^* h^k_\mu = -\frac{\partial}{\partial \xi^\mu}(h^k_\nu \delta\xi^\nu).$$

The third equation concerning δh^k_μ represents that h^k_μ is a covariant vector for the transformation (11).

Now we have introduced the ξ-system into the flat space-time quite artificially. Hence any physical principle should not depend on the choice of ξ-system. Therefore for the change (11) of the choice of ξ-system, we should have

$$\delta I = \int_\Omega \left\{\delta^* \mathfrak{L} + \frac{d}{d\xi^\mu}(\mathfrak{L}\delta\xi^\mu)\right\} (d\xi)^4 \equiv 0.$$

In the same way as (6) has been derived, we obtain

$$\delta^*\mathfrak{L}+\frac{d}{d\xi^\mu}(\mathfrak{L}\delta\xi^\mu)\equiv 0, \tag{12}$$

where

$$\delta^*\mathfrak{L}=\frac{\partial\mathfrak{L}}{\partial Q_A}\delta^*Q_A+\frac{\partial\mathfrak{L}}{\partial Q_{A,\mu}}\delta^*Q_{A,\mu}+\frac{\partial\mathfrak{L}}{\partial h^k_\mu}\delta^*h^k_\mu.$$

(12) can be written as follows:

$$[\mathfrak{L}]_{Q_A}\delta^*Q_A+\frac{\partial}{\partial\xi^\mu}\left(\frac{\partial\mathfrak{L}}{\partial h^k_\mu}\right)h^k_\nu\delta\xi^\nu-\frac{\partial}{\partial\xi^\mu}\left\{\frac{\partial\mathfrak{L}}{\partial h^k_\mu}h^k_\nu\delta\xi^\nu+\mathfrak{T}^\mu_\nu\delta\xi^\nu\right\}\equiv 0, \tag{13}$$

where

$$\mathfrak{T}^\mu_\nu=h^k_\nu\mathfrak{T}^\mu_k=\mathfrak{P}^{A,\mu}Q_{A,\nu}-\delta^\mu_\nu\mathfrak{L}.$$

Substituting (13) into the expression of δI, and assuming that $\delta\xi$ vanishes on the boundary hypersurface of Ω, we obtain

$$\int_\Omega\left\{-[\mathfrak{L}]_{Q_A}Q_{A,\nu}+\frac{\partial}{\partial\xi^\mu}\left(\frac{\partial\mathfrak{L}}{\partial h^k_\mu}\right)h^k_\nu\right\}\delta\xi^\nu(d\xi)^4\equiv 0.$$

Now taking into account that $\delta\xi$ can be taken arbitrarily inside Ω, we can put

$$\frac{\partial}{\partial\xi^\mu}\left(\frac{\partial\mathfrak{L}}{\partial h^k_\mu}\right)\equiv[\mathfrak{L}]_{Q_A}Q_{A,\nu}h^\nu_{;k}. \tag{14}$$

Substituting from (14) back into (13) we obtain

$$\frac{\partial}{\partial\xi^\mu}\left\{\left(\frac{\partial\mathfrak{L}}{\partial h^k_\mu}h^k_\nu+\mathfrak{T}^\mu_\nu\right)\delta\xi^\nu\right\}\equiv 0. \tag{15}$$

Putting

$$\mathfrak{R}^\mu_\nu\equiv\frac{\partial\mathfrak{L}}{\partial h^k_\mu}h^k_\nu+\mathfrak{T}^\mu_\nu$$

and noticing that every derivatives of $\delta\xi$ of any rank can be taken as independent, we can derive the following identities from (15)

$$\frac{\partial\mathfrak{R}^\mu_\nu}{\partial\xi^\mu}\equiv 0 \qquad \mathfrak{R}^\mu_\nu\equiv 0,$$

that is

$$-\frac{\partial\mathfrak{L}}{\partial h^k_\mu}h^k_\nu\equiv\mathfrak{T}^\mu_\nu$$

or

$$\frac{\partial\mathfrak{L}}{\partial h^k_\mu}\equiv-\mathfrak{T}^\mu_k. \tag{16}$$

Therefore (14) becomes

$$\frac{\partial \mathfrak{T}_k^\mu}{\partial \xi^\mu} \equiv -[\mathfrak{L}]_{Q_A} Q_{A,k}. \tag{14'}$$

Substituting from these relations into (10), we shall obtain (7) again.

If we make use of the field equation, we can obtain from (14) and (7) the following conservation laws.

$$\frac{\partial \mathfrak{T}_k^\mu}{\partial \xi^\mu} = 0 \tag{17}$$

$$\frac{\partial}{\partial \xi^\mu}\{\mathfrak{P}^{A,\mu}\delta Q_A - \mathfrak{T}_k^\mu \delta x_k\} = 0. \tag{18}$$

Example 1. Translation group.

This group is defined by $\delta x^k = \epsilon^k = \text{const.}$, $\delta Q_A = 0$, $\delta \xi^\mu = 0$. In this case (18) becomes

$$\frac{\partial}{\partial \xi^\mu}(\mathfrak{T}_k^\mu) = 0$$

This is nothing but (17). Integrating this with $(d\xi)^4$ we obtain the four constants of motion

$$J_k = \int_{\xi^0=\text{const}} \mathfrak{T}_k^0 d\xi^1 d\xi^2 d\xi^3 = \int_{\xi^0=\text{const}} \mathfrak{T}_k^0 d\vec{\xi}. \tag{19}$$

This is the energy-momentum four vector of the system considered The vector character for the Lorentz transformation can be easily seen from the expression (19).

Now some elementary calculations show

$$D h_k^0 d\vec{\xi} = d\sigma_k, \tag{20}$$

where $d\sigma_k$ is the element of the hypersurface $\xi^0 = \text{const.}$ of which the components are given by

$$d\sigma_1 = dx^2 dx^3 dx^0 \quad d\sigma_2 = dx^3 dx^0 dx^1 \quad d\sigma_3 = dx^0 dx^1 dx^2 \quad d\sigma_0 = dx^1 dx^2 dx^3.$$

Therefore (19) can be transformed as follows:

$$J_k = \int_{\xi^0=\text{const}} T_k^l d\sigma_l. \tag{19}$$

This expression agrees with those given by Schwinger.

Example 2. Lorentz group.

This group is defined by

$$\delta \xi^\mu = 0, \quad \delta x^k = \epsilon^{km} x_m = \frac{1}{2}\epsilon^{mn}(\delta_m^k x_n - \delta_n^k x_m)$$

$$\delta Q_A = \frac{1}{2} D_{A,mn}^B \epsilon^{mn} Q_B \quad D_{A,mn}^B = -D_{A,nm}^B \quad \epsilon^{mn} = -\epsilon^{nm}.$$

Substituting from these into (18), we obtain

$$\frac{\partial}{\partial\xi^\mu}\{\mathfrak{P}^{A\mu}D^B_{A,mn}Q_B-\mathfrak{T}^\mu_m x_n+\mathfrak{T}^\mu_n x_m\}=0. \tag{21}$$

Integrating this with respect to $(d\xi)^4$, we can obtain a constant antisymmetric tensor

$$J_{mn}=\int_{\xi^0=\text{const}}\mathfrak{P}^A D^B_{A,mn}Q_B-(\mathfrak{T}^0_m x_n-\mathfrak{T}^0_n x_m)\}d\vec{\xi} \tag{22}$$

where $\qquad \mathfrak{P}^A=\mathfrak{P}^{A,0}$

(22) may be written, using (20), as follows:

$$J_{mn}=\int_{\xi^0=\text{const}}\{P^{Al}D^B_{A,mn}Q_B-(T^l_m x_n-T^l_n x_m)\}d\sigma_l\,. \tag{23}$$

This is the total angular momentum tensor of the system considered.

Example 3. Consider the following infinitesimal transformation.

$$\delta x^k=0,\ \ \delta\xi^\mu=0,\ \ \delta Q_A=C_{A,L}(\xi,Q)\lambda^L+C^\mu_{A,L}(\xi,Q)\frac{\partial\lambda^L}{\partial\xi^\mu} \tag{24}$$

where $\lambda^L(\xi)\,(L=1,2,\cdots\gamma)$ are γ arbitrary functions of ξ. Now let us assume I to be invariant for this transformation. Then from (7′), it follows

$$[\mathfrak{L}]_{Q_A}C_{AL}\lambda^L-\frac{\partial}{\partial\xi^\mu}([\mathfrak{L}]_{Q_A}C^\mu_{AL})\lambda^L+\frac{\partial}{\partial\xi^\mu}\Big\{(\mathfrak{P}^{A\mu}C_{AL}+[\mathfrak{L}]_{Q_A}C^\mu_{AL})\lambda^L$$
$$+\mathfrak{P}^{A\mu}C^\nu_{AL}\frac{\partial\lambda^L}{\partial\xi^\nu}\Big\}\equiv 0. \tag{25}$$

Following the method of consideration by which we have derived the identities (14) and (15) from the original identity (13), we can derive from (24) the following identities provided that λ^L vanishes on the boundary surface of integral domain Ω, and also it can be taken completely arbitrarily inside Ω;

$$[\mathfrak{L}]_{Q_A}C^{AL}-\frac{\partial}{\partial\xi^\mu}\{[\mathfrak{L}]_{Q_A}C^\mu_{AL}\}\equiv 0, \tag{25}$$

and

$$\frac{\partial}{\partial\xi^\mu}\Big\{\mathfrak{R}^\mu_L\lambda^L+\mathfrak{R}^{\mu,\nu}_L\frac{\partial\lambda^L}{\partial\xi^\nu}\Big\}\equiv 0, \tag{26}$$

where

$$\mathfrak{R}^\mu_L=\mathfrak{P}^{A,\mu}C_{AL}+[\mathfrak{L}]_{Q_A}C^\mu_{AL}$$
$$\mathfrak{R}^{\mu,\nu}_L=\mathfrak{P}^{A\mu}C^\nu_{AL}\,.$$

(25) means that the N field equations are not independent to each other, but there are γ functional relations like (25). From (26) we obtain an expression

$$\int\limits_{\dot{\xi}^0=\text{const}}\left\{\left(\mathfrak{R}_L^{\dot{0}}-\frac{d}{d\xi^{\bar{\nu}}}\mathfrak{R}_L^{\dot{0}\bar{\nu}}\right)\lambda^L+\mathfrak{R}_L^{\dot{0}\dot{0}}\frac{\partial\lambda^L}{\partial\xi^{\dot{0}}}\right\}d\xi$$

which is constant with respect to $\xi^{\dot{0}}$.

Differentiating this with respect to $\xi^{\dot{0}}$, and noticing that λ^L, $\frac{\partial\lambda^L}{\partial\xi^{\dot{0}}}$ and $\frac{\partial^2\lambda^L}{\partial\xi^{\dot{0}2}}$ can qe considered as independent to each other, we get the following identities:

$$\frac{d}{d\xi^{\dot{0}}}\left(\mathfrak{R}_L^{\dot{0}}-\frac{d}{d\xi^{\bar{\nu}}}\mathfrak{R}_L^{\dot{0}\bar{\nu}}\right)\equiv 0. \tag{27.1}$$

$$-\mathfrak{R}_L^{\dot{0}}+\frac{d}{d\xi^{\bar{\nu}}}\mathfrak{R}_L^{\dot{0}\bar{\nu}}\equiv\frac{d}{d\xi^{\dot{0}}}\mathfrak{R}_L^{\dot{0}\dot{0}} \tag{27.2}$$

$$\mathfrak{R}_L^{\dot{0}\dot{0}}\equiv\mathfrak{P}^A C_{AL}^{\dot{0}}\equiv 0. \tag{27.3}$$

(27.3) means that the N-$\mathfrak{P}^A$ can not be considered to be independent. For this reason, we meet with some formal difficulties if we shall try to formulate the whole theory in Hamilton's canonical formalism in which all the $\mathfrak{P}^B$ and Q_A are to be taken as independent.

§ 3. Quantum Theory of Fields.

We shall consider in this paragraph those cases in which all the $\mathfrak{P}^A$ can be taken as independent. The exceptional case as in the example 3 of the preceding paragraph will be treated in the next paper.

Let us postulate the following commutation relations ($C. R.$) on any hyper-surface $\xi^{\dot{0}}=$const.,

$$[\mathfrak{P}^A(\vec{\xi},\xi^{\dot{0}}),\ Q_B(\vec{\xi}',\xi^{\dot{0}})]=-i\delta^A_B\delta(\vec{\xi}-\vec{\xi}') \tag{28}$$ [11]

where

$$\delta(\vec{\xi}-\vec{\xi}')=\delta(\xi^{\dot{1}}-\xi^{\dot{1}\prime})\delta(\xi^{\dot{2}}-\xi^{\dot{2}\prime})\delta(\xi^{\dot{3}}-\xi^{\dot{3}\prime}).$$

All the other commutators are put equal to zero.

In order to show the equivalence of (28) with the ordinary $C. R.$ referring to any Lorentz system, we integrate (28)

$$\int\limits_{\dot{\xi}^0=\text{const}}[\mathfrak{P}^A(\vec{\xi}',\xi^{\dot{0}}),\ Q_B(\vec{\xi},\xi^{\dot{0}})]d\vec{\xi}'=\int\limits_{\dot{\xi}^0=\text{const}}[P^{Ak}(x'),\ Q_B(x)]d\sigma_k'=-i\delta^A_B.$$

This is equivalent to Schwinger's expression. Now if we shall be able to show the invariance of (28) for any ξ-transformation, then we may prove the equivalence of (28) with the ordinary $C. R.$ by performing the ξ-transformation successively until the $\xi^{\dot{0}}$-surface will coincide with the plane $x^0=$const. The invariance of (28) for any infinitesimal ξ-transformation will be shown in the later part of this paragraph.

Now we take as the Hamiltonian of this system the following quantity

$$\bar{\mathfrak{H}}=\int_{\xi^0=\mathrm{const}}\mathfrak{H}\,d\vec{\xi}=\int_{\xi^0=\mathrm{const}}\mathfrak{T}_0^0\,d\vec{\xi}.$$

Here we assume that the ambiguities of the orders of $\mathfrak{P}$ and Q appearing in $\mathfrak{T}_0^0$ have been removed by the usual manner of symmetrization (making hermitian). The field equations are given by

$$\frac{dA}{d\xi^0}=i[\bar{\mathfrak{H}},\ A] \tag{29}$$

where A is any function of $\mathfrak{P}$ and Q only, and does not depend on ξ^0 explicitly. In our case where $\mathfrak{L}$ has the restricted form (2), (29) formally agrees with those of the classical theory. It should be noticed that our Hamiltonian $\bar{\mathfrak{H}}$ is not a constant of motion because it depends on ξ^0 explicitly.

Now our purpose in this paragraph is to establish the quantum formalism when the physical system is invariant for the transformation groups already stated in § 2.

In the first place let us consider the transformation (5). In the classical theory we obtain a constant of motion from (18)

$$G=\int_{\xi^0=\mathrm{const}}\{\mathfrak{P}^A\delta Q_A-\mathfrak{T}_k^0\delta x^k\}\,d\vec{\xi} \tag{30}$$

provided that the field equations are satisfied. Next we shall investigate this constancy in the quantum theory. To do this, let us observe the following equations,

$$i[G,Q_A]=\delta Q_A-Q_{A,k}\delta x^k$$

$$i[G,\mathfrak{P}^A]=-\mathfrak{P}^B\frac{\partial\delta Q_B}{\partial Q_A}-\frac{\partial}{\partial\xi^\mu}(\mathfrak{P}^A\Delta\xi^\mu)+\mathfrak{P}^{A\mu}\frac{\partial\Delta\xi^0}{\partial\xi^\mu} \tag{31}$$

where

$$\Delta\xi^\mu=h_k^\mu\delta x^k.$$

From the first equation of (31), we can put the following equation if x's are independent parameters;

$$i[G,Q_A]=\delta^*Q_A. \tag{31'}$$

But from this standpoint, δP^{Ak} is given by

$$\delta P^{Ak}=-\frac{\partial\delta x^m}{\partial x^m}P^{A,k}-P^{B,k}\frac{\partial\delta Q_B}{\partial Q_A}+P^{A,l}\frac{\partial\delta x_k}{\partial x^l}. \tag{32}$$

(As to the orders of $\mathfrak{P}$ and Q, we assume the symmetrization procedure has been performed. Hereafter we shall always make this assumption if necessary). And then

$$\delta^*\mathfrak{P}^{A\mu} = \delta\mathfrak{P}^{A\mu} - \frac{\partial\mathfrak{P}^{A\mu}}{\partial x^k}\delta x^k$$
$$= -\frac{\partial\delta Q_B}{\partial Q_A}\mathfrak{P}^{B\mu} - \frac{\partial\mathfrak{P}^{A\mu}}{\partial\xi^\nu}\Delta\xi^\nu. \tag{33}$$

Therefore we can *not* claim

$$i[G, \mathfrak{P}^A] = \delta^*\mathfrak{P}^A.$$

But from the equation

$$Dh_k^0\delta^* P^{Ak} = -\frac{\partial(\mathfrak{P}^{A\mu}\Delta\xi^\nu)}{\partial\xi^\nu} - \mathfrak{P}^{B\mu}\frac{\partial\delta Q_B}{\partial Q_A} + \mathfrak{P}^{A\nu}\frac{\partial\Delta\xi^\mu}{\partial\xi^\nu}, \tag{34}$$

we can put

$$i[G, \mathfrak{P}]^A = iDh_k^0[G, P^{Ak}] = Dh_k^0\delta^* P^{Ak}.$$

Combining this with (31) we obtain

$$\delta^* P^A = i[G, P^A]$$
$$\delta^* Q^A = i[G, Q^A] \tag{35}$$

where

$$P^A = P^{A,0}.$$

From this result we see that the generating operator for this transformation (5) is G when we take P^A, Q^A as independent field quantities, and x as independent parameters. Hence, if A is any function of P^A, Q_A and $h_\mu^k(x)$, then $\delta^* A$ is given by

$$\delta^* A = i[G, A] + \frac{\partial A}{\partial h_\mu^k}\delta^* h_\mu^k. \tag{36}$$

Next we consider the transformation (11). If we put

$$K \equiv -\int_{\xi^0=\mathrm{const}} \mathfrak{T}_\mu^0\delta\xi^\mu d\vec{\xi}, \tag{37}$$

then we can easily obtain the following relations:

$$i[K, Q_A] = -Q_{A\mu}\delta\xi^\mu \tag{38.1}$$

$$i[K, \mathfrak{P}^A] = -\frac{\partial}{\partial\xi^\mu}(\mathfrak{P}^A\delta\xi^\mu) + \mathfrak{P}^{A\mu}\frac{\partial\delta\xi^0}{\partial\xi^\mu}. \tag{38.2}$$

From (38.1), if we take ξ as independent parameters, we get

$$i[K, Q^A] = \delta^* Q^A. \tag{38.1'}$$

In this case where ξ are independent parameters, $\delta\mathfrak{P}^{A\mu}$ takes the following form:

$$\delta\mathfrak{P}^{A\mu} = -\mathfrak{P}^{A\mu}\frac{\partial\delta\xi^\nu}{\partial\xi^\nu} + \mathfrak{P}^{A\nu}\frac{\partial\delta\xi^\mu}{\partial\xi^\nu}.$$

Hence we can put

$$i[K, \mathfrak{P}^A] = \delta^* \mathfrak{P}^A \tag{38.2'}$$

Combining (38,1') and (38.2'), we can claim for any quantity $A = A(\mathfrak{P}, Q, h^k_\mu)$ the following relation

$$\delta^* A = i[K, A] + \frac{\partial A}{\partial h^k_\mu} \delta^* h^k_\mu. \tag{39}$$

From this result we know that the generating operator for this transformation (11) is K when we take $\mathfrak{P}^A$, Q_A as independent field quantities and ξ as independent parameters.

Now, in the special case where $\delta\xi^\mu = \epsilon^\mu = \text{const.}$ we get the following relations from (39):

$$\frac{dA}{d\xi^\mu} = \frac{\partial A}{\partial \xi^\mu} + i\left[\int_{\xi^0=\text{const}} \mathfrak{T}^0_\mu d\xi,\ A\right]. \tag{40}$$[12]

Especially if we put $\mu=0$, then (40) is a generalization of (29) itself.

In the same way, if we put δx^k in G equal to ϵ^k and $\delta Q^A = 0$, then G becomes as follows:

$$G = -\epsilon^k J_k = -\epsilon^k \int_{\xi^0=\text{const}} \mathfrak{T}^0_k d\xi$$

and (36) is transformed into

$$\frac{\partial A}{\partial x^k} = i[J_k A] + \frac{\overset{\circ}{\partial} A}{\partial x^k}. \tag{41}$$

Now (35) and (38') can be transformed into forms of unitary transformation. That is, in (35), we can define an unitary operator U as follows:

$$U = 1 + iG$$

because of the hermitian character of G which is possible in most cases. Making use of this U, (35) is transformed into the form

$$P'^A(x) = U P^A(x) U^{-1}$$

$$Q'_A(x) = U Q^A(x) U^{-1}.$$

In the case of (38'), K should be used instead of G.

From these expressions it can be easily shown that the C.R (28) is invariant for any infinitesimal ξ-transformation.

Our next task is to prove the constancy of G with respect to ξ^0. Let us consider any quantity $A = A(P, Q)$. From the formula

$$\frac{d}{dx^k} \delta^* A = \delta^* \frac{dA}{dx^k},$$

we get the following relations

$$\left(\frac{d}{dx^k}\delta^* - \delta^*\frac{d}{dx^k}\right)A = i\left[\frac{dG}{dx^k},\ A\right] = 0,$$

where we have made use of (41) and the fact that J_k does not depend on x explicitly. From the above result we obtain

$$\frac{dG}{dx^k} = 0 \quad \frac{dG}{d\xi^{\dot{0}}} = h^k_{\dot{0}}\frac{dG}{dx^k} = 0. \tag{42}$$

This means that G is a constant of motion, and at the same time expresses the invariancy of G for any ξ-transformation.

Next let us investigate this situation about K. In this case we know also the same relation

$$\frac{d}{d\xi^\mu}\delta^* A = \delta^*\frac{dA}{d\xi^\mu}$$

for an arbitrary function $A = A(\mathfrak{P}, Q)$ if we take $\mathfrak{P}$, Q as independent field quantities and ξ as independent parameters. Following the same line of reasoning as in the previous case, we can derive the relation

$$\left(\delta^*\frac{d}{d\xi^\mu} - \frac{d}{d\xi^\mu}\delta^*\right)A = -i\left[\left(\frac{dK}{d\xi^\mu} - \mathring{\delta}^* J_\mu\right),\ A\right] = 0,$$

where

$$J_\mu = \int \mathfrak{T}^{\dot{0}}_\mu d\vec{\xi}$$

and $\mathring{\delta}^* J_\mu$ expresses the variation of J_μ by virtue of its explicit dependence on ξ. From the above result, we get

$$\frac{dK}{d\xi^\mu} = \mathring{\delta}^* J_\mu . \tag{43}$$

In particular, for $\mu = \dot{0}$, (43) becomes

$$\frac{dK}{d\xi^{\dot{0}}} = \mathring{\delta}^* J_{\dot{0}} = \mathring{\delta}^* \bar{\mathfrak{H}} \tag{43'}$$

that is K is not a constant of motion as easily seen. Further we can see that $\bar{\mathfrak{H}}$ is not a constant as was already stated, because $\bar{\mathfrak{H}}$ is a special example for K.

Now we can show the covariance of the field equation (29) for any ξ-transformation if we make use of (43′). The transformation character of $\bar{\mathfrak{H}}$ for ξ-transformation (11) is given by

$$\bar{\mathfrak{H}}(\xi^{\dot{0}}) \rightarrow \bar{\mathfrak{H}}'(\xi^{\dot{0}}) = \bar{\mathfrak{H}}(\xi^{\dot{0}}) + i[K\bar{\mathfrak{H}}] + \mathring{\delta}^*\bar{\mathfrak{H}} \tag{44}$$

using the general formula (39). And an arbitrary quantity $A = A(\mathfrak{P}, Q)$ is transformed by (11) as follows:

$$A(\mathfrak{P}, Q) \rightarrow A(\mathfrak{P}', Q') = UA(\mathfrak{P}, Q)U^{-1} \equiv A'(\xi^{\dot{0}}).^{13)}$$

Taking notice of the definition for U, we can calculate $dA'/d\xi^{\dot{0}}$ in the following manner

$$\frac{dA'}{d\xi^{\dot{0}}} = i\left[\left(\frac{dK}{d\xi^{\dot{0}}} + \mathfrak{H} + i[K, \bar{\mathfrak{H}}]\right), A'\right].$$

Substituting from (44) and (43′) into the above result we get

$$\frac{dA'}{d\xi^{\dot{0}}} = i[\bar{\mathfrak{H}}'(\xi^{\dot{0}}), A'].$$

In conclusion we see as stated above that the quantum theory of fields can be formulated in a covariant form for any ξ-transformation and Lorentz transformation.

We have been so far assuming the hypersurface $\xi^{\dot{0}}$=const. to be space-like. But we want to make a few remarks about this assumption. Contrary to the assumption mentioned above, let us assume the hypersurface σ on which $\xi^{\dot{0}}$ is a constant to be time like.[14] Then we know from (28) that $\mathfrak{P}(A)$ and $Q(B)$ are commutative with each other, if A and B are two world points on σ (see Fig. 1).

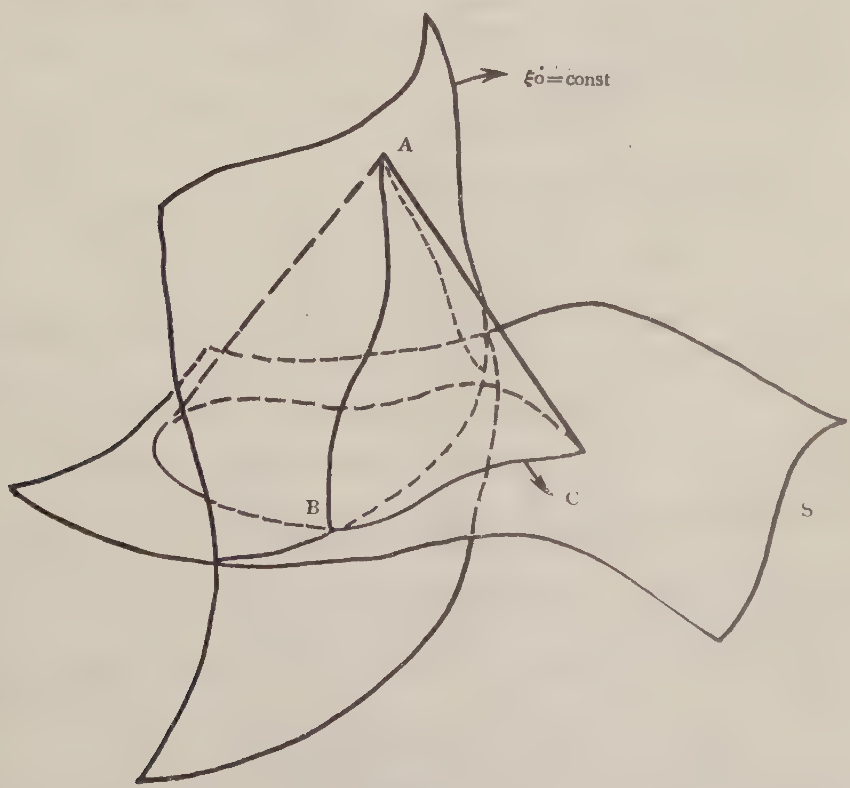

Fig. 1.

Consider a light cone having the point A as its vertex and also an arbitrary space-like surface S containing the point B. Let us call the intersection between the above light cone and the surface S by C. Remembering that the characteristic of the field equations is a usual light cone, $\mathfrak{P}(A)$ and $Q(A)$ are functional of $\mathfrak{P}$ and Q on and inside of C. Hence from $[Q(A), Q(B)]=0$, $Q(B)$ becomes commutative with $\mathfrak{P}$ and Q on and inside of C. Especially $Q(B)$ becomes commutative with $\mathfrak{P}(B)$ contradicting to the postulates (28). But if we assume the surface σ to be space-like, then this contradiction disappears as easily seen. Therefore the necessity of the space-like character of the surface σ is not clear a priori but results from the field equations.

§ 4. Super-many Time Theory.

Although we can show the covariance of the field equations for any ξ-transformation, it is not easy to see at a glance their covariant character because of the complication of the behavior of $\bar{\mathfrak{H}}$ for this transformation. To make clear this covariance it is useful to reformulate the whole theory in the form of super-many time theory.

For this purpose we restrict the ξ-transformation in the following manner: $\xi^{\bar{\mu}}$ is unaltered, while $\xi^{\dot{0}}$ is infinitesimally changed at the neighbourhood of the world point P on the surface σ on which $\xi^{\dot{0}}=C=\text{const}$. That is, let us consider the infinitesimal transformation

$$\delta\xi^{\dot{0}}(\xi)=\lambda(\xi)\delta(\vec{\xi}-\vec{\xi}_P)$$

where λ is an infinitesimal function of ξ.
Corresponding to this, $\delta_P^{\times}x^k$ is given by

$$\delta_P^{\times}x^k(\xi)=-h^k_{\dot{0}}(\xi)\lambda(\xi)\delta(\vec{\xi}-\vec{\xi}_P).$$

Consider another transformation of the same nature at a second world point Q on σ which is given by

$$\delta\xi^{\dot{0}}(\xi)=\lambda'(\xi)\delta(\vec{\xi}-\vec{\xi}_Q).$$

Then the following relation is obtained by some elementary calculations:

$$(\delta_Q^{\times}\delta_P^{\times}-\delta_P^{\times}\delta_Q^{\times})x^k(\xi)=h^k_{\dot{0}}(\xi)\delta(\vec{\xi}-\vec{\xi}_P)\delta(\vec{\xi}-\vec{\xi}_Q)\left\{\frac{\partial\lambda(\xi)}{\partial\xi^{\dot{0}}}\lambda'(\xi)-\frac{\partial\lambda'(\xi)}{\partial\xi^{\dot{0}}}\lambda(\xi)\right\}.$$

Hence if we assume that λ and λ' are independent on $\xi^{\dot{0}}$ near the surface σ, then all these transformations constitute a commutative group.

Now the generating operator K corresponding to these transformations is given by

$$K(P)=-\mathfrak{T}^{\dot{0}}_{\dot{0}}\delta\xi^{\dot{0}}d\xi)_P=-\underset{D}{\mathfrak{T}^{\dot{0}}_{\dot{0}}}\cdot D\delta\xi^{\dot{0}}d\xi)_P.$$

If we choose $\delta\overset{\circ}{\xi}{}^{0}$ to be negative or zero everywhere on σ, then $-D\delta\overset{\circ}{\xi}{}^{0}d\vec{\xi})_P$ is a four dimensional positive volume element dw^4 at P spanned by $\delta\overset{\circ}{\xi}{}^{0}$ and $d\vec{\xi}$. Hence $K(P)$ is written as follows:

$$K(P)=\frac{\overset{\circ}{\mathfrak{T}}{}_{0}^{0}}{D}dw^4)_P\equiv H(P)\,dw^4_P.$$

Now let us call the transformed surface σ', on which the value of $\xi'^{\dot{0}}$ is equal to that of $\xi^{\dot{0}}$ on σ, i.e. $\xi'^{\dot{0}}=C$.

Then it should be noticed that the surface σ' juts out towards the future near the point P as compared with σ on account of our choice for $\delta\xi^{\dot{0}}$. (cf. Fig. 2).[15)]

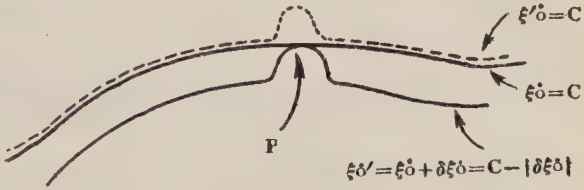

Fig. 2

Let us integrate $Q_A(\xi)$ over the surface σ with regard to $d\vec{\xi}$, keeping the value of $\xi^{\dot{0}}$ to be always equal to C, then this integrated quantity can be considered as a functional of σ. Hence we can put

$$Q_A[\sigma]=\int_{\sigma}Q_A(\xi,\,\xi^{\dot{0}}=C)\,d\vec{\xi}$$

Then for the transformation considered above, the following relation is derived by virtue of (38.1′);

$$Q_A[\sigma']-Q_A[\sigma]=\int_{\xi^{\dot{0}\prime}=C}Q'_A(\vec{\xi}'\xi^{\dot{0}\prime}=C)\,d\vec{\xi}'-\int_{\xi^{\dot{0}}=C}Q_A(\vec{\xi},\,\xi^{\dot{0}}=c)\,d\vec{\xi}$$

$$=\int_{\sigma}\delta^{*}Q_A(\xi,\,\xi^{0}=C)\,d\vec{\xi}=i\,dw^4_P[H(P),\;Q_A[\sigma]].$$

Dividing the both sides of this equation with $dw^4)_P$ and recalling the definition of functional derivatives, we can put

$$\frac{\delta Q_A[\sigma]}{\delta\sigma_P}=i[H(P),\,Q_A[\sigma]].$$

In quite the same way, we get the following equation for $\mathfrak{P}^A[\sigma]$;

$$\frac{\delta\mathfrak{P}^A[\sigma]}{\delta\sigma_P}=i[H(P),\,\mathfrak{P}^A[\sigma]].$$

In general, it holds

$$\frac{\delta A}{\delta\sigma_P}=i[H(P),\,A]+\frac{\overset{\circ}{\delta}A}{\delta\sigma_P}\tag{45}$$ [16)]

where A is any function of $\mathfrak{P}[\sigma]$ and $Q[\sigma]$ and may depend on σ in some cases,

and $\mathring{\delta}A/\delta\sigma_P$ is the variation of A due to its explicit dependence on σ.

Now since our transformations are commutative, the following relation must be satisfied;

$$\frac{\delta^2 A}{\delta\sigma_P\delta\sigma_Q}=\frac{\delta^2 A}{\delta\sigma_Q\delta\sigma_P}.\text{[17]}$$

Hence, substituting (45) into this equation and making use of Jacobi's identity, we get

$$\left[\left\{\left(\frac{\mathring{\delta}H(Q)}{\delta\sigma_P}-\frac{\mathring{\delta}H(P)}{\delta\sigma_Q}\right)-i[H(Q), H(P)]\right\}, A\right]=0$$

provided that A is independent on σ explicitly. This means

$$\frac{\mathring{\delta}H(P)}{\delta\sigma_Q}-\frac{\mathring{\delta}H(Q)}{\delta\sigma_P}=i[H(P), H(Q)]. \tag{46}$$

Thus we have obtained two types of the field equations. The relation of these two types can be easily shown as below. From the definition of δ^* we can write $\delta Q/\delta\sigma$ in the following form:

$$\frac{\delta Q_A[\sigma]}{\delta\sigma_P}=\lim_{\Delta w_P\to 0}\frac{1}{\Delta w_P}\int_\sigma \delta^* Q_A d\vec{\xi}=\frac{1}{D}\left.\frac{\partial Q_A}{\partial\xi^0}\right)_P.$$

On the other hand it holds that

$$i[H(P), Q_A[\sigma]]=i\frac{1}{D_P}[\bar{\mathfrak{H}}, Q_A(P)] \tag{47}$$

on account of the fact that $\mathfrak{H}$ does not contain the derivatives of $\mathfrak{P}^A$.
Hence from (45) we get

$$\frac{\partial Q_A(\xi)}{\partial\xi^0}=i[\bar{\mathfrak{H}}, Q_A(\xi)].$$

As to $\mathfrak{P}^A$, the equality like (47) does not hold, because $\bar{\mathfrak{H}}$ contains the derivatives of Q. That is,

$$i[H(P), \mathfrak{P}^A[\sigma]]=\frac{1}{D_P}\left.\frac{\partial\mathfrak{L}}{\partial Q_A}\right)_P$$

where $\mathfrak{L}$ is to be considered as a function of Q_A and $\mathfrak{P}^A$. Contrary to the case of Q, it holds that

$$\frac{\delta\mathfrak{P}^A[\sigma]}{\delta\sigma_P}=\frac{1}{D_P}\frac{\partial\mathfrak{P}^{A\mu}}{\partial\xi^\mu}$$

where we have made use of the assumption that $\delta\xi^0$ is independent on ξ^0. Combining the above two results, we get

$$\frac{\partial \mathfrak{P}^A(\xi)}{\partial \xi^{\dot 0}} = i[\mathfrak{H}, \mathfrak{P}^A(\xi)].$$

Our last task in this paragraph is to rewrite the expression $H(P)$. Let us denote by n_k the k-th component of a unit normal of the surface σ at P towards future direction. Then n_k is given by

$$n_k = -\frac{h^{\dot 0}_k}{\sqrt{-r^{\dot 0 \dot 0}}}.$$

The minus sign of the right hand side corresponds to our choice of the ξ-system.[15] If the ξ-system is orthogonal, (at least near the surface σ) i.e.

$$r^{\bar\mu \dot 0} = r_{\bar\mu \dot 0} = 0,$$

then we can put

$$n_k = -\frac{h^{\dot 0 k}}{\sqrt{-r^{\dot 0 \dot 0}}} = h^k_{\dot 0}\sqrt{-r_{\dot 0 \dot 0}}\,.$$

Hence $H(P)$ can be transformed into the following form;

$$H(P) = \left.\frac{\mathfrak{T}^{\dot 0}_{\dot 0}}{D}\right)_P = h^{\dot 0}_l h^k_{\dot 0} T^l_k = -n_l n^k T^l_k. \tag{48}$$

§ 5. Interaction Representation.

Consider a system of many fields interacting with each other and assume H of this system to have the following form:

$$H = H^0 + V \tag{49}$$

where H^0 corresponds to a sum of Hamiltonians of each field without interactions, and V to a density of interaction energy between these fields.

Now we consider the following transformation for any quantity $\boldsymbol{A}[\sigma]$

$$\boldsymbol{A}[\sigma] \to A[\sigma] = U[\sigma]\boldsymbol{A}[\sigma]U^{-1}[\sigma]$$

where $U[\sigma]$ is a unitary operator and a functional of σ. (In this section any field quantity of Heisenberg picture will be denoted by a bold face letter).

If we take such $U[\sigma]$ as to satisfy the following equation

$$i\frac{\delta U[\sigma]}{\delta \sigma_P}U^{-1}[\sigma] = V(P, \sigma) = U[\sigma]\boldsymbol{V}(p)U^{-1}[\sigma], \tag{50}$$

then we get the following equation

$$\frac{\delta A[\sigma]}{\delta \sigma_P} = i[H^0_P, A[\sigma]]. \tag{51}$$

These A's are called as the interaction representation of $\boldsymbol{A}$.

Here we must investigate the integrability of (51). Differentiate $H^0(P)$ by σ at a point Q, we get

$$\frac{\delta H_P^0}{\delta\sigma_Q}=\frac{\mathring{\delta} H_P^0}{\delta\sigma_Q}+i[H_Q^0, H_P^0]$$

$$=\left[\frac{\delta U}{\delta\sigma_Q}U^{-1}\ H_P^0\right]+U\left\{i[\boldsymbol{H}_Q, \boldsymbol{H}_P^0]+\frac{\mathring{\delta}\boldsymbol{H}_P^0}{\delta\sigma_Q}\right\}U^{-1}$$

i.e.

$$\frac{\mathring{\delta} H_P^0}{\delta\sigma_Q}=U\left(\frac{\mathring{\delta}\boldsymbol{H}_P}{\delta\sigma_Q}-\frac{\mathring{\delta}\boldsymbol{V}_P}{\delta\sigma_Q}\right)U^{-1}.$$

Interchanging P and Q and subtracting one from the other we obtain

$$\frac{\mathring{\delta} H_P^0}{\delta\sigma_Q}-\frac{\mathring{\delta} H_Q^0}{\delta\sigma_P}=i[H_P, H_Q]-U\left(\frac{\mathring{\delta}\boldsymbol{V}_P}{\delta\sigma_Q}-\frac{\mathring{\delta}\boldsymbol{V}_Q}{\delta\sigma_P}\right)U^{-1}. \tag{52}$$

If we make the interaction parameters tend to zero we obtain

$$\frac{\mathring{\delta} H_P^0}{\delta\sigma_Q}-\frac{\mathring{\delta} H_Q^0}{\delta\sigma_P}=i[H_P^0, H_Q^0]. \tag{53}$$

Hence the remaining part of (52) runs as follows:

$$U\left(\frac{\mathring{\delta}\boldsymbol{V}_P}{\delta\sigma_Q}-\frac{\mathring{\delta}\boldsymbol{V}_Q}{\delta\sigma_P}\right)U^{-1}=i[V_P, V_Q]+i[V_P, H_Q^0]+i[H_P^0, V_Q], \tag{54}$$

because the C. R. is independent whether the interaction parameters are zero or not hence (52) also holds when these parameters do not vanish.

If we take into account the following relation

$$\frac{\delta V_P}{\delta\sigma_Q}=i[H_Q^0, V_P]+U\frac{\mathring{\delta}\boldsymbol{V}_P}{\delta\sigma_Q}U^{-1},$$

we obtain, the following equation from (54)

$$\frac{\delta V_Q}{\delta\sigma_Q}-\frac{\delta V_Q}{\delta\sigma_P}=i[V_P, V_Q]. \tag{55}$$

(55) is the integrability condition of (50), and (53) is that of (51).

Next we introduce a new quantity $A[\xi, \sigma]$ by the following equation

$$\boldsymbol{A}(\xi)\rightarrow A[\xi, \sigma]=U[\sigma]\boldsymbol{A}(\xi)U^{-1}[\sigma]. \tag{56}$$

If a point P is apart from the point Q having coordinates ξ, then

$$\frac{\delta A[\xi,\sigma]}{\delta\sigma_P}=-i[V_P, A[\xi, \sigma]]=0. \tag{57}$$

On the contrary, if P is in the neighbourhood of Q, (57) does not in general vanish. This means that $A[\xi, \sigma]$ depends on the form of σ at the point Q.

Now (50) can be transformed into an integral equation as follows:

$$U[\sigma]=1-i\int_{\sigma^0}^{\sigma} V[\xi', \sigma']U[\sigma'](dw')^4 \tag{58}$$

where σ^0 is a fixed surface and $U[\sigma]$ is taken to be unity as σ coincides with σ^0. Further the point ξ' is taken to lie on σ'. From (58) the change of U corresponding to a change of the value C of $\dot{\xi^0}$ at the surface σ is determined by the following equation

$$\frac{\partial U}{\partial \dot{\xi^0}}=-i\int_{\dot{\xi^0}=C} DV[\xi, \xi^0=C, \sigma]d\xi\cdot U[\sigma]. \tag{58'}$$

In (58'), the surface σ contained in U and V is not deformed, but translated towards the direction for which $\dot{\xi^0}$ increases keeping the form of σ's unaltered. From this standpoint, U can be considered as a function of $\dot{\xi^0}$ rather than a functional of σ. Further from the same standpoint, $A[\xi, \sigma]$ can be considered as a function of $(\vec{\xi}, \dot{\xi^0})$ only. Then the differential equation satisfied by the $A(\xi)$ runs as follows:

$$\frac{\partial A(\xi)}{\partial \dot{\xi^0}}=i[\bar{\mathfrak{H}}^0, A(\xi)]. \tag{59}$$

Next we denote the state function in Heisenberg picture by Φ_0 and that of interaction representation by $\Psi[\sigma]$, the relation of these is given by

$$\Psi[\sigma]=U[\sigma]\Phi_0. \tag{60}$$

And the generalized Schroedinger equation is derived from (50) as follows:

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma_P}=V[P, \sigma]\Psi[\sigma]. \tag{61}$$

The condition of integrability is satisfied on account of (55).

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(to be continued)

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8) We assume D to be a definite function, not to be vanishing anywhere in the space-time.

9) Here we assume the N field equations to be functionally independent to each other.

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11) The hyper-surface $\xi^0=$const. is assumed to be space-like. The necessity of this condition will be discussed in § 3.

12) $\mathring{\partial} A/\partial\xi^\mu$ means derivative of A due to its explicit dependence on ξ. This notation will be used hereafter.

13) We assume that A may be a function of $\mathfrak{P}$, Q and the derivatives of these with regard to $\xi^{\bar{\mu}}$ but must not contain the derivatives about $\xi^{\dot{0}}$.

14) It is unnecessary to consider such a case in which the surface $\xi^{\dot{0}}=$const. is partially space-like, and at the same time time-like in another part on account of footnote. (8).

15) We take the direction for which $\xi^{\dot{0}}$ increases to coincide with that of x^0, hence $h^{\dot{0}}_{0}$ and $h^{0}_{\dot{0}}$ are positive.

16) In this case, the independent field quantities are $\mathfrak{P}$ and Q as already stated in § 3.

17) Strictly speaking, we know only $(\delta^{*}_{\dot{P}}\delta^{\kappa}_{Q}-\delta^{\kappa}_{\dot{Q}}\delta^{\kappa}_{P})A=0$. If we make use of this relation, and further of the assumption that $\delta\xi^{\dot{0}}$ being independent on $\xi^{\dot{0}}$, then we can obtain the commutativity of functional derivatives.

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Radiative Corrections to Decay Processes. I.

——*The Meson Decay.*——

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§ 1. Introduction.

In order to see if the idea of mass and charge renormalizations is applicable to the cases of several particles interacting with each other, the second order radiative corrections to the proper life of meson or to the β-decay probabilities of nuclei have been investigated. The results for the first case are reported here and those for the other will appear in the succeeding paper.

According to the Yukawa model, a charged meson decays spontaneously into an electron and a neutrino at a certain rate, w_0, through the direct interaction among them. This process is unavoidably accompanied by a sudden acceleration of a unit electronic charge, which, according to the classical electrodynamics, causes emission of electromagnetic waves, whose intensities distribute almost uniformly over the spectral regions from zero frequency up to a certain upper limit.[1] Quantum-mechanically interpreted, this implies that the rate, $\delta_1 w$, of meson decay that is accompanied by real photon emission diverges in the infrared region——an infrared catastrophe (Feer[2]).

According to a general consideration[3], such as by Bloch and Nordsieck, the infrared divergence of $\delta_1 w$ can be cancelled by that of the other radiative correction to the rate of radiationless decay, $\delta_0 w$, which is due to the emission and reabsorption of virtual photons. This correction, however, brings in an ultraviolet divergence owing to the infinite degrees of freedom of high frequency photons. It is shown that the ultraviolet divergence encountered by $\delta_0 w$ can be removed, for the most part, by virtue of the mass renormalizations, and the remaining part with the aid of the renormalization of coupling constant, g, between the meson and electron-neutrino fields. The state of affairs is similar to that in the well-known problem of the radiative correction for electron scattering.

The calculation is performed with the use of the method of contact transformation in the covariant formalism developed by Tomonaga[4], Schwinger[5], Dyson[6], and Feynman[7].

For simplicity, the case of scalar meson coupling with electron-neutrino fields through the Hamiltonian of scalar type is treated, the neutrino mass being assumed to be zero. The result obtained is applicable directly to π-μ decay, but some modifications are necessary in applying it to the decay of μ-meson, as two neutrinos seem to be emitted simultaneously in this case.

§ 2. Formulation.

In the covariant formalism, our system is described by the generalized Schrödinger equation of the form

$$\left\{H(x)+V(x)-i\frac{\partial}{\partial\sigma(x)}\right\}\Psi[\sigma]=0, \tag{2.1}$$

where $H(x)$ and $V(x)$ denote the Hamiltonians which account respectively for the interaction of the radiation field with the electron and the meson, and for the coupling between meson and electron-neutrino fields.

Let $u(x)$, $\psi(x)$ and $\varphi(x)$ denote the wave functions of meson, electron and neutrino in this order, $u^*(x)$, $\bar{\psi}(x)$ and $\bar{\varphi}(x)$ their Hermitic conjugate or adjoint operators, then $V(x)$ is given by

$$V(x)=g(\bar{\varphi}(x)\psi(x)u(x)+u^*(x)\bar{\psi}(x)\varphi(x)) \tag{2.2}$$

with g as the coupling constant.

According to Koba, Tati and Tomonaga,[4] the interaction Hamiltonian between the electron and radiation fields is given by

$$H_e(x)=ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \tag{2.3}$$

while, following Kanesawa and Tomonaga,[5] that between the meson and radiation fields by

$$H_m(x)=H_m^{(1)}(x)+H_m^{(2)}(x),$$

with $$H_m^{(1)}(x)=ieA_\mu(x)u^*(x)\Gamma_\mu u(x) \tag{2.4}$$

and $$H_m^{(2)}(x)=e^2(\delta_{\mu\nu}+N_\mu N_\nu)A_\mu(x)A_\nu(x)u^*(x)u(x),$$

where N_μ is the normal at the point x on the surface σ, $A_\mu(x)$ denotes the 4-potential of the radiation fields, and it is understood that

$$u^*(x)\Gamma_\mu u(x)=u^*(x)\partial_\mu u(x)-(\partial_\mu u^*(x))u(x). \tag{2.5}$$

Hence

$$H(x)=H_e(x)+H_m(x)-\delta m\bar{\psi}(x)\psi(x)-\delta\varkappa^2u^*(x)u(x), \tag{2.6}$$

where the counter-self-energy terms are already included in order that we may start from the stage in which the mass renormalizations have been done for electron and meson (See Appendix).

The wave functions are subjected to the wave equations for free fields, and to the usual ordan-Pauli's commutation relations, of which the mass parameters are to be regarded as the observed ones.

With the Schwinger's formula on the transition probability[9] applied to our problem, the proper life of the meson inclusive of the radiative corrections is, to the g^2 approximation, given by the inverse of

$$w=\int d^3x \int (dx')(1|V_F^*(x')V_F(x)|1), \tag{2.7}$$

where $V_F(x)$ is constructed in accordance with

$$V_F(x)=S[\infty]S^{-1}[\sigma]V(x)S[\sigma] \tag{2.8}$$

with the aid of the unitary transformation function $S[\sigma]$, which is a solution of the equation

$$\left\{H(x)-i\frac{\partial}{\partial\sigma(x)}\right\}S[\sigma]=0, \tag{2.9}$$

subjected to the initial condition

$$S[-\infty]=1, \tag{2.10}$$

the bracket $(1|\quad|1)$ denotes the diagonal matrix element with respect to the initial state, and the x integration is to be taken over the whole three dimensional surface perpendicular to the time axis of the reference system in which the meson before decay is at rest, while the x' integration over the whole four-dimensional space-time.

$V_F(x)$ is expressed in ascending powers of e as

$$V_F(x)=V(x)+V_F^{(1)}(x)+V_F^{(2)}(x)+\cdots \tag{2.11}$$

with

$$\begin{aligned}V_F^{(1)}(x)&=-i\int (dx')P(V(x),H_e(x')+H_m^{(1)}(x'))\\&=iegu(x)\bar\varphi(x)\int(dx')S_+(x-x')\gamma_\mu\psi(x')A_\mu(x')\\&\quad-ieg\bar\varphi(x)\psi(x)\int(dx')\Delta_+(x-x')\Gamma''_\mu u(x')\cdot A_\mu(x')\\&\quad+\text{conj.}\end{aligned} \tag{2.12}$$

$$\begin{aligned}V_F^{(2)}(x)=&-i\int (dx')P(V(x),H_m^{(2)}(x')-\delta m\bar\psi(x')\psi(x')-\delta\kappa^2u^*(x')u(x'))\\&+\frac{(-i)^2}{2}\int(dx')\int(dx'')P(V(x),H_e(x')+H_m^{(1)}(x'),H_e(x'')+H_m^{(1)}(x'')),\end{aligned} \tag{2.13}$$

where $P(\quad)$'s denote the Dyson's brackets.[6)]

To the e^2 approximation, the total rate of meson decay w can now be split into three parts as

$$w=w_0+\delta_0 w+\delta_1 w \tag{2.14}$$

with

$$w_0=\int d^3x\int (dx')(1|V(x)V(x')|1), \tag{2.15}$$

$$\delta_0 w=\int d^3x\int (dx')(1|V(x)V_F^{(2)}(x')+V_F^{(2)*}(x)V(x')|1), \tag{2.16}$$

and

$$\delta_1 w=\int d^3x\int (dx')(1|V_F^{(1)*}(x)V_F^{(1)}(x')|1), \tag{2.17}$$

where w_0 represents the rate of meson decay without radiative corrections, and $\delta_0 w$ describes the correction to the rate of radiationless decay due to the virtual photon processes, while $\delta_1 w$ accounts for decay that is accompanied by single photon emission.

Extracting from $V_F^{(2)}(x)$, with the aid of the Dyson's rules, the part which gives contribution to $\delta_0 w$, one may replace $V_F^{(2)}(x)$ in (2.16) by

$$< V_F^{(2)}(x) > =g\overline{\varphi}(x)\psi'^{(2)}(x)u(x)+g\overline{\varphi}(x)\psi(x)u'^{(2)}(x)+g\overline{\varphi}(x)K[\psi(x)u(x)] \tag{2.18}$$

with

$$\psi'^{(2)}(x)=\int (dx')S_+(x-x')\left\{ie^2\int (dx'')\gamma_\mu S_+(x'-x'')\gamma_\mu\psi(x'')D_+(x'-x'')-\delta m\psi(x')\right\}, \tag{2.19}$$

$$\begin{aligned} u'^{(2)}(x)=\int (dx')\Delta_+(x-x')\Big\{&\delta\kappa^2 u(x') \\ &+ie^2\int (dx'')(\Gamma''_\mu\Delta_+(x'-x'')\Gamma''_\mu+\delta_{\mu\mu}\delta(x'-x''))u(x'')\cdot D_+(x'-x'')\Big\}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &K[\psi(x)u(x)] \\ &=-ie^2\int (dx')\int (dx'')S_+(x-x')\gamma_\mu\psi(x')D_+(x'-x'')\cdot\Delta_+(x-x'')\Gamma''_\mu u(x''). \end{aligned} \tag{2.21}$$

These integrals are evaluated with the substitution of the following Fourier integral representations for D_+-functions,

$$D_+(x)=\frac{1}{(2\pi)^4}\int (dk)\frac{e^{ikx}}{k^2-i\epsilon},$$

$$S_+(x)=\frac{1}{(2\pi)^4}\int(dp)\frac{(i\gamma p-m)e^{ipx}}{p^2+m^2-i\epsilon}, \tag{2.22}$$

and

$$\Delta_+(x)=\frac{1}{(2\pi)^4}\int(dq)\frac{e^{iqx}}{q^2+\varkappa^2-i\epsilon},$$

in which m and $\varkappa$ are the masses respectively of electron and of meson, and the limit $\epsilon\longrightarrow+0$ is understood.

According to Dyson,[6] $\psi'^{(2)}(x)$ can be transformed to the momentum integral

$$\psi'^{(2)}(x)=-Z^{(2)}\int(dp)\frac{i\gamma p-m}{p^2+m^2-i\epsilon}(i\gamma p+m)e^{ipx}\psi(p), \tag{2.23}$$

where $\psi(p)$ is the Fourier amplitude of $\psi(x)$, satisfying $(i\gamma p+m)\psi(p)=0$. Though this expression is indeterminate between 0 and $Z^{(2)}\psi(x)$ according as $(i\gamma p+m)$ is operated on the right or on the left, $\psi'^{(2)}(x)$ should be renormalized as

$$\psi'^{(2)}(x)=\frac{1}{2}Z^{(2)}\psi(x) \tag{2.24}$$

so as to reserve the unitarity of S matrix.

The second order renormalization factor $Z^{(2)}$ is evaluated by Karplus and Kroll[10] as follows.

$$Z^{(2)}=\frac{2ie^2}{(2\pi)^4}\int_0^1 udu\int(dk)\frac{k^2-2m^2(2-2u-u^2)}{(k^2+m^2u^2-i\epsilon)^3}, \tag{2.25}$$

or with the k and u integrations performed,

$$Z^{(2)}=\frac{e^2}{(2\pi)^2}\left\{\log\frac{m}{2k_0}-\frac{1}{2}\log\frac{k_\infty+\sqrt{k_\infty^2+m^2}}{m}+\frac{1}{8}\right\}, \tag{2.26}$$

k_0 and k_∞ representing respectively the lower and the upper cut-off frequencies of virtual photons.

In a similar way, $u'^{(2)}(x)$ can be expressed in terms of the Fourier integrals as

$$u'^{(2)}(x)=-\frac{ie^2}{(2\pi)^4}\int(dq)\frac{e^{iqx}}{q^2+\varkappa^2-i\epsilon}(q^2+\varkappa^2)u(q)\int(dk)\int_0^1 du\cdot$$

$$\cdot\left\{\frac{(3-2u)k^2+(1+2u)\varkappa^2u^2+3u(1-u)(q^2+\varkappa^2)}{(k^2+\varkappa^2u^2+u(1-u)(q^2+\varkappa^2)-i\epsilon)^3}\right.$$

$$\left.+\int_0^1 dv\cdot 2\varkappa^2u(1-u)\frac{2(2-u)k^2+(4+u)\varkappa^2u^2+2u(1-u)(2-u)v(q^2+\varkappa^2)}{(k^2+\varkappa^2u^2+u(1-u)v(q^2+\varkappa^2)-i\epsilon)^4}\right\}, \tag{2.27}$$

which also is indeterminate between 0 and $Y^{(2)}u(x)$ with

$$Y^{(2)}=\frac{-ie^2}{(2\pi)^4}\int(dk)\int_0^1 du\left\{\frac{(3-2u)k^2+u(1-2u)(4-3u)\varkappa^2}{(k^2+\varkappa^2u^2-i\epsilon)^2}+\frac{6\varkappa^4u^4(1-u)}{(k^2+\varkappa^2u^2-i\epsilon)^4}\right\}$$

$$=\frac{e^2}{(2\pi)^2}\left\{\log\frac{\varkappa}{2k_0}+\log\frac{k_\infty+\sqrt{k_\infty^2+\varkappa^2}}{\varkappa}+\frac{1}{8}\right\},\tag{2.28}$$

since the factor $(q^2+\varkappa^2)$ operating on $u(q)$ gives 0, while operating on $1/(q^2+\varkappa^2-i\epsilon)$ gives 1. But, on the same reason as before, one must take

$$u'^{(2)}(x)=\frac{1}{2}Y^{(2)}u(x).\tag{2.29}$$

Inserting (2.22) into (2.21), and performing the x', x'', p and q integrations, we have

$$K[\psi(x)u(x)]=-\frac{ie^2}{(2\pi)^4}\int(dk)\int_0^1 udu\int_{-1}^1 dv.$$

$$\frac{k^2-u^2\lambda^2-2(1-u)(m^2+\varkappa^2-\square)+u(1-v)\varkappa^2-u(1+v)m(\gamma\partial)}{(k^2+\lambda^2u^2-i\epsilon)^3}(\psi(x)u(x)),\tag{2.30}$$

with
$$\lambda^2=\frac{1+v}{2}m^2+\frac{1-v}{2}\varkappa^2-\frac{1-v^2}{4}\square.\tag{2.31}$$

With the k and u integrations performed, (2.30) becomes

$$K[\psi(x)u(x)]=L(\square,\gamma\partial)(\psi(x)u(x)),\tag{2.32}$$

with
$$L(\square,\gamma\partial)=\frac{e^2}{4(2\pi)^2}\left\{\log\frac{k_\infty+\sqrt{k_\infty^2+m^2}}{m}+\log\frac{k_\infty+\sqrt{k_\infty^2+\varkappa^2}}{\varkappa}\right.$$
$$-(m^2+\varkappa^2-\square)\left(\log\frac{\sqrt{\varkappa m}}{2k_0}+\frac{1}{4}\right)F_0(\square)-\frac{m^2+\varkappa^2-\square}{2}G(\square)$$
$$\left.+\frac{\varkappa^2}{4}(F_0(\square)-F_1(\square))-\left(\frac{m^2}{4}+\frac{m(\gamma\partial)}{2}\right)(F_0(\square)+F_1(\square))\right\},\tag{2.33}$$

where

$$F_n(\square)=\int_{-1}^1 dv\frac{v^n}{\lambda^2}$$

and

$$G(\square)=\int_{-1}^1 dv\frac{1}{\lambda^2}\log\frac{\lambda^2}{\varkappa m}.\tag{2.34}$$

As will be shown later, for the purpose of evaluating $\delta_0 w$, the whole expression $L(\square,\gamma\partial)$ is unnecessary, but

$$L(0,0)=\frac{e^2}{4(2\pi)^2}\left\{\log\frac{k_\infty+\sqrt{k_\infty^2+m^2}}{m}+\log\frac{k_\infty+\sqrt{k_\infty^2+\varkappa^2}}{\varkappa}-2\frac{\varkappa^2+m^2}{\varkappa^2-m^2}\log\frac{\varkappa^2}{m^2}\right.$$
$$\left.\cdot\log\frac{\sqrt{\varkappa m}}{2k_0}-\frac{1}{2}\frac{\varkappa^2+m^2}{\varkappa^2-m^2}\log\frac{\varkappa^2}{m^2}-2\frac{\varkappa^2m^2}{(\varkappa^2-m^2)^2}\log\frac{\varkappa^2}{m^2}+\frac{\varkappa^2+m^2}{\varkappa^2-m^2}\right\}\tag{2.35}$$

suffices. In deriving (2.35) from (2.33), the relations

$$F_0(0)=\frac{2}{\kappa^2-m^2}\log\frac{\kappa^2}{m^2},$$

$$F_1(0)=\frac{2}{\kappa^2-m^2}\left(\frac{\kappa^2+m^2}{\kappa^2-m^2}\log\frac{\kappa^2}{m^2}-2\right), \tag{2.36}$$

and

$$G(0)=0$$

have been used.

§ 3. Decay Probabilities.

For definiteness we assume the meson considered to be of positive electric charge.

Let us first evaluate w_0. In accordance with (2.2) and (2.15), we have

$$w_0=g^2\int d^3x\int(dx')(1|u^*(x)\bar{\psi}(x)\varphi(x)\bar{\varphi}(x')\psi(x')u(x')|1), \tag{3.1}$$

of which the matrix element can be factorized into

$$-(1|u^*(x)u(x')|1)S_p[S_n^{(+)}(x-x')S^{(-)}(x'-x)] \tag{3.2}$$

with the use of the relations on the vacuum values

$$<\varphi(x)\bar{\varphi}(x')>_0=-iS_n^{(+)}(x-x')=-\frac{1}{(2\pi)^3}\int_{r_0>0}(dr)(i\gamma r)\delta(r^2)e^{ir(x-x')}$$

and

$$<\bar{\psi}_\alpha(x)\psi_\beta(x')>_0=-iS_{\beta\alpha}^{(-)}(x'-x)=\frac{1}{(2\pi)^3}\int_{p_0<0}(dp)\,(i\gamma p-m)_{\beta\alpha}\delta(p^2+m^2)e^{ip(x-x')}.$$

Let $q_\mu=(0,0,0,i\kappa)$ denote the 4-momentum of the meson considered, then

$$(1|u^*(x)u(x')|1)=(1|u^*(q)u(q)|1)e^{iq(x'-x)} \tag{3.4}$$

where $u^*(q)$ and $u(q)$ are the Fourier amplitudes of $u^*(x)$ and $u(x)$. The matrix element of $u^*(q)u(q)$ can be determined by virtue of the relation which states that there exists the positive meson at rest:

$$e=-i\int d^3x(1|-ieu^*(q)e^{-iqx}\Gamma_4 e^{iqx}u(q)|1)=2e\kappa V_3(1|u^*(q)u(q)|1), \tag{3.5}$$

where V_3 represents the volume of the whole three dimensional space. Hence

$$(1|u^*(x)u(x')|1)=\frac{1}{2\kappa V_3}e^{-iq(x-x')}. \tag{3.6}$$

Inserting (3.2) and (3.6) into (3.1), and performing the x integration, we get

$$w_0=-\frac{g^2}{2\kappa}\int(dx')e^{iqx'}S_p[S_n^{(+)}(-x')S^{(-)}(x')], \tag{3.7}$$

which, after the x' integration with the use of the Fourier representations of $S_n^{(+)}(-x')$ and $S^{(-)}(x')$, becomes

$$w_0=-\frac{g^2}{2\kappa(2\pi)^2}\int_{p_0>0}(dp)\int_{r_0>0}(dr)\delta(q-r+p)\delta(r^2)\delta(p^2+m^2)\cdot S_p[(i\gamma p-m)(i\gamma r)], \tag{3.8}$$

where $-p$ and r may be interpreted as the 4-momenta of the decay products, respectively of a positron and of a neutrino, and the factor $\delta(q-r+p)$ describes the energy-momentum conservation.

Performing the p and r integrations, we obtain

$$w_0=\frac{g^2}{4\pi}\cdot\frac{(\kappa^2-m^2)^2}{2\kappa^3}. \tag{3.9}$$

Next, we evaluate $\delta_0 w$, which is, according to (2.16), (2.18), (2.24), (2.27) and (2.30), given by

$$\delta_0 w=\int d^3x\int(dx')(1|V(x)<V_F^{(2)}(x')>+<V_F^{(2)}(x)>V(x')|1) \tag{3.10}$$

with

$$<V_F^{(2)}(x)>=g\overline{\varphi}(x)\left\{\frac{1}{2}Y^{(2)}+\frac{1}{2}Z^{(2)}+L(\Box,\gamma\partial)\right\}(\psi(x)u(x))+\text{conj.} \tag{3.11}$$

In this expression the operator $\Box$ means, properly speaking, minus the norm of the sum of the 4-momenta of meson and of electron, which, however, may as well be reinterpreted as that of the 4-momentum of neutrino by taking account of the energy-momentum conservation which is satisfied in the real transition, and vanishes owing to the assumption of the null neutrino mass. Similarly one may put $(\gamma\partial)=0$ in (3.11) for the purpose of evaluating $\delta_0 w$. Thus (3.10) reduces to

$$\delta_0 w=\{Y^{(2)}+Z^{(2)}+2L(0,0)\}\int d^3x\int(dx')(1|V(x)V(x')|1), \tag{3.12}$$

from which, remembering (2.15), (2.24), (2.26), and (2.32), we obtain

$$\frac{\delta_0 w}{w_0}=\frac{e^2}{(2\pi)^2}\left\{\frac{3}{2}\log\frac{k_\infty+\sqrt{k_\infty^2+\kappa^2}}{\kappa}-\left(\frac{\kappa^2+m^2}{\kappa^2-m^2}\log\frac{\kappa^2}{m^2}-2\right)\log\frac{\sqrt{\kappa m}}{2k_0}\right.$$
$$\left.-\frac{1}{4}\frac{\kappa^2+m^2}{\kappa^2-m^2}\log\frac{\kappa^2}{m^2}-\frac{\kappa^2m^2}{(\kappa^2-m^2)^2}\log\frac{\kappa^2}{m^2}+\frac{1}{2}\frac{\kappa^2+m^2}{\kappa^2-m^2}+\frac{1}{4}\right\}. \tag{3.13}$$

Finally we must evaluate $\delta_1 w$ in accordance with (2.12) and (2.17). The method of calculation is similar to the case of w_0. The intermediate results obtained by performing the coordinate space integrations is

$$\delta_1 w = \frac{e^2 g^2}{2\varkappa(2\pi)^5}\int_{k_0>0}(dk)\int_{p_0<0}(dp)\int_{r_0>0}(dr)\,\delta(k^2)\,\delta(p^2+m^2)\,\delta(r^2)\,\delta(p+q-k-r)$$

$$\cdot\left\{\frac{1}{((p-k)^2+m^2)^2}S_p[(i\gamma(p-k)-m)\gamma_\mu(i\gamma p-m)\gamma_\mu(i\gamma(p-k)-m)\,(i\gamma r)]\right.$$

$$-\frac{2}{((p-k)^2+m^2)\,((q-k)^2+\varkappa^2)^2}S_p[(i\gamma p-m)\,i\gamma(2q-k)\,(i\gamma(p-k)-m)\,(i\gamma r)]$$

$$\left.-\frac{1}{((q-k)^2+\varkappa^2)^2}(2q-k)^2 S_p[(i\gamma p-m)\,(i\gamma r)]\right\}, \tag{3.14}$$

where k, $-p$, and r represent the 4-momenta of respectively the photon, positron, and neutrino. In the derivation of this expression, the formula on the vacuum expectation value of $A_\mu(x)A_\nu(x')$,

$$\langle A_\mu(x)A_\nu(x')\rangle_0 = i\,\delta_{\mu\nu}D^{(+)}(x-x') = \frac{1}{(2\pi)^3}\delta_{\mu\nu}\int_{k_0>0}(dk)\,\delta(k^2)\,e^{ik(x-x')}, \tag{3.15}$$

has been used.

After performing the integrations over the directions of the momenta, we get

$$\delta_1 w = \int_0^{\frac{\varkappa-m}{2}} w(k)\,dk, \tag{3.16}$$

with

$$w(k)\,dk = w_0\left\{\left(\frac{\varkappa^2+m^2}{\varkappa^2-m^2}-\frac{2\varkappa k}{\varkappa^2-m^2}+\frac{2\varkappa^2k^2}{(\varkappa^2-m^2)^2}\right)\log\frac{\varkappa^2-2\varkappa k}{m^2}\right.$$

$$\left.-2\left(1-\frac{2\varkappa k}{\varkappa^2-m^2}\right)\left(1+\frac{\varkappa k^2}{(\varkappa^2-m^2)\,(\varkappa-2k)}\right)\right\}\frac{dk}{k}, \tag{3.17}$$

k representing the photon frequency.

$w(k)dk$ describes the spectral rate of meson decay that is accompanied by the emission of a photon of frequency between k and $k+dk$. Consequently the spectral intensity of the radiation emitted in the decay is given by $w(k)\,k\,dk$.

As a measure of the magnitude of the rate of radiative decay, let us estimate, following Feer[2], the ratio of the mean energy emitted in radiation per unit time to the mean energy available for the process per unit time, which is essentially the rate of radiationless decay times the meson rest energy $w_0\varkappa$:

$$R = \int_0^{\frac{\varkappa-m}{2}}\frac{w(k)\,k}{w_0\varkappa}dk$$

$$=\frac{1}{137\pi}\left\{\left(\frac{\varkappa^2+m^2}{\varkappa^2-m^2}\,\frac{2\varkappa-m}{2\varkappa}-\frac{1}{4}\,\frac{\varkappa-m}{\varkappa+m}+\frac{1}{12}\,\frac{(\varkappa-m)^3}{\varkappa(\varkappa^2-m^2)}-\frac{1}{6}\,\frac{\varkappa^4}{(\varkappa^2-m^2)^2}\right)\log\frac{\varkappa}{m}\right.$$

$$\left.+\frac{\varkappa-m}{\varkappa}-\frac{2\varkappa^2-\varkappa m+m^2}{2\varkappa(\varkappa+m)}-\frac{1}{36}\,\frac{\varkappa(\varkappa-2m)}{(\varkappa+m)^2}+\frac{1}{6}\,\frac{\varkappa^3(\varkappa-m)}{(\varkappa^2-m^2)^2}\right\} \tag{3.18}$$

$$=\begin{cases} 0.81\% & \text{for } x=200\text{m}, \qquad (3.19)\\ 0.29\% & \text{for } x=286,\ m=210 \text{ electron mass}, \qquad (3.20)\end{cases}$$

of which the latter is applicable to the $\pi-\mu$ decay.

Performing the momentum integrations of (3.14), we obtain

$$\frac{\delta_1 w}{w_0}=\frac{e^2}{(2\pi)^2}\left\{\left(\frac{x^2+m^2}{x^2-m^2}\log\frac{x^2}{m^2}-2\right)\log\frac{x-m}{2k_0}-\frac{2x^4+2x^3m-7x^2m^2}{8(x^2-m^2)^2}\log\frac{x^2}{m^2}\right.$$
$$\left.+\frac{17x^4-16x^3m-31x^2m^2+28xm^3}{8(x^2-m^2)^2}-\frac{x^2+m^2}{x^2-m^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{m}{x}\right)^n\right\}. \qquad (3.21)$$

On comparing this expression with (3.13), it will be observed that the infrared divergence encountered by $\delta_1 w$ can be exactly cancelled by that occurring in $\delta_0 w$. It is to be noticed here the second order renormalization factors $Y^{(2)}$ and $Z^{(2)}$ of the wave functions are indispensable for the perfect cancellation of the infrared catastrophe.

$\delta_0 w$ involves an ultraviolet divergence, which has not its origin in the electromagnetic masses because the mass renormalizations are already performed. In order to get a finite radiative correction to the proper life of meson, this divergence must be removed. The removal of this divergence may be attained with the aid of the renormalization of g, for instance, by reinterpreting,

$$g_{ob}=g\left(1+\frac{3}{4}\log\frac{k_\infty+\sqrt{k_\infty^2+x^2}}{x}\right) \qquad (3.22)$$

as the observed coupling constant between meson and electron-neutrino fields. The g renormalization, however, can not be uniquely determined, for one may as well push finite terms of $\frac{1}{2}\delta_0 w$ by an arbitrary amount into the observed g. There is no criterion to determine how much of the finite terms of $\frac{1}{2}\delta_0 w$ should be amalgamated, in the retinue of the divergent term, into g_{ob}, in contrast to the case of the charge renormalization.

If g is renormalized provisionally in accordance with (3.22), the total specific radiative correction to the rate of meson becomes

$$\frac{\delta_0 w+\delta_1 w}{w_0}=\frac{e^2}{(2\pi)^2}\left\{\left(\frac{x^2+m^2}{x^2-m^2}\log\frac{x^2}{m^2}-2\right)\log\frac{x-m}{\sqrt{xm}}\right.$$
$$-\frac{4x^4+2x^3m+x^2m^2-2m^4}{8(x^2-m^2)^2}\log\frac{x^2}{m^2}+\frac{21x^4-16x^3m-31x^2m^2+28xm^3-4m^4}{8(x^2-m^2)^2}$$
$$\left.+\frac{1}{4}-\frac{x^2+m^2}{x^2-m^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{m}{x}\right)^n\right\}, \qquad (3.23)$$

which is simplified to

$$\frac{\delta_0 w+\delta_1 w}{w_0}=\frac{1}{137\pi}\left\{\left(\log\frac{x}{m}-1\right)^2+\frac{15}{8}-\frac{\pi^2}{6}\right\} \quad (x\gg m) \qquad (3.24)$$

in case where $x\gg m$, and is estimated as

$$\frac{\delta_0 w+\delta_1 w}{w_0}=\begin{cases} 5.0\% & \text{for} \quad 300\text{m}, \\ 4.1\% & \text{for} \quad 200\text{m}. \end{cases} \qquad (3.25)$$

The expression (2.23) holds also for the $\pi-\mu$ decay, provided that the π and μ mesons obey respectively the Klein-Gordon and the Dirac equations. The numerical value for this case amounts to

$$(\delta_0 w+\delta_1 w)/w_0=-0.42\% \qquad (3.26)$$

with the masses of the π and μ mesons as respectively 286 and 210 times the electron mass.

Appendix.

In this section we shall make a remark on the renormalization of the meson mass.

According to Baba and Kanesawa,[11] the self-energy operator of the scalar meson does not reduce to the form

$$\delta x^2 \ u^*(x)u(x), \qquad (\text{A. }1)$$

but contains an extra term,

$$A_s L_s(x)=-A_s(\partial_\mu u^*(x)\cdot\partial_\mu u(x)+x^2 \ u^*(x)u(x)), \qquad (\text{A. }2)$$

proportional to the Lagrangian of the free meson field with a logarithmically divergent coefficient A_s, if it is defined by the one particle part of the operator

$$-\frac{i}{2}\int_{-\infty}^{\sigma}(dx')[H_m^{(1)}(x),\ H_m^{(1)}(x')]+H_m^{(2)}(x), \qquad (\text{A. }3)$$

which is the analogue of the definition used by Tati and Tomonaga[1] and by Schwinger[2] for derivation of the self-energy operator of electron

$$\delta m\bar{\psi}(x)\psi(x). \qquad (\text{A. }4)$$

The term of the form (A. 2) destroys the integrability of the generalized Schrödinger equation (2.1), if it is added, with its sign reversed, to the Hamiltonian $H_m(x)$ as a counter-self-energy term. To avoid this difficulty, Baba and Kanesawa propose the modification of the Lagrangian of the meson field.

It is, however, not only preferable but feasible to perfom the self-consistent subtraction of the divergent self-energy without the aid of the Lagrangian renormalization which will make computations of higher order effects much cumbersome.

For this purpose, we have only to take

$$-\delta x^2 u^*(x)u(x) \qquad (\text{A. }5)$$

as the counter-self-energy term, as in (2.6) of the text, discarding the term $A_s \cdot L_s(x)$, which gives no contribution to the expectation value of the self-energy but otherwise harmful to the integrability of the generalized Schrödinger equation.

It can be shown that the omission of the term $A_s L_s(x)$ from the counter-self-energy term introduces nothing wrong with the radiative corrections.

As a typical example, let us examine the steadiness of the one meson state. For this purpose we construct the one meson part of the S matrix, which, to the c^2 approximation, reads

$$<S[\infty]>_{10}=1-i\int(dx)\{-\delta\varkappa^2 u^*(x)u(x)$$

$$-ic^2\int(dx')u^*(x)(\Gamma'_\mu\Delta_+(x-x')\Gamma_\mu'+\delta_{\mu\mu}\delta(x-x'))u(x')\cdot D_+(x-x')\}. \qquad \text{(A. 6)}$$

The second term in the bracket of this expression is just the same operator as that defined by the one particle part of (A. 3), and in fact, with the x' integration performed, takes the form

$$\delta\varkappa^2 u^*(x)u(x)+A_s L_s(x), \qquad \text{(A. 7)}$$

with

$$\delta\varkappa^2=-3iD_+(0)+2A_s\varkappa^2$$

and

$$A_s=-\frac{ic^2}{(2\pi)^4}\int(dk)\int_0^1 du\frac{(2-u)k^2+2\varkappa^2u^2}{(k^2+\varkappa^2u^2-i\epsilon)^3}$$

$$=\frac{c^2}{(2\pi)^2}\left\{\frac{3}{4}\log\frac{k_\infty+\sqrt{k_\infty^2+\varkappa^2}}{\varkappa}+\frac{1}{4}\right\}, \qquad \text{(A. 8)}$$

whence

$$S[\infty]>_{10}=1-iA_s\int_{-\infty}^{\infty}(dx)L_s(x). \qquad \text{(A. 9)}$$

The space-time integral of $L_s(x)$ is expressed in terms of the Fourier integral as

$$\int_{-\infty}^{\infty}(dx')L_s(x)=-\int(dq)u^*(q)(q^2+\varkappa^2)u(q), \qquad \text{(A. 10)}$$

which vanishes owing to the relation either $(q^2+\varkappa^2)u(q)=0$ or $u^*(q)(q^2+\varkappa^2)=0$. Thus

$$<S[\infty]>_{10}=1, \qquad \text{(A. 11)}$$

from which the steadiness of the one-meson state follows.

As this example shows, the self-energy term (A. 7) always appears in the form integrated over the whole space-time, whenever it is contained in radiative corrections. Therefore, the effects of the term $A_s L(x)$ on the radiative corrections automatically vanish. This is the reason why the term $A_s L_s(x)$ does not miss its counter term in the Hamiltonian $H(x)$.

We have determined, in the text, the second order renormalization factor $\frac{1}{2}\Gamma^{(2)}$

of the wave function for meson following the Dyson's method. In this connection, also the Schwinger's method is to be referred to.

In order to evaluate the function $u'^{(2)}(x)$ given by (2.20) following the latter method, we first transform it, with the help of the integration by parts with respect to x', into the form

$$u'^{(2)}(x)=\int(dx')\Delta_+(x-x')\{\delta\varkappa^2 u(x')-\eta(x')\}, \tag{A. 12}$$

with

$$\eta(x)=-ie^2\int(dx')\{D_+(x-x')\cdot(2\partial_\mu\Delta_+(x-x')\Gamma_\mu'+\delta_{\mu\mu}\delta(x-x'))$$
$$+\partial_\mu D_+(x-x')\cdot\Delta_+(x-x')\Gamma_\mu'\}u(x'). \tag{A. 13}$$

With the x' integration performed, (A. 13) reduces to

$$\eta(x)=\delta\varkappa^2 u(x). \tag{A. 14}$$

Hence we obtain

$$u'^{(2)}(x)=0 \tag{A. 15}$$

in one hand.

On the other hand, $u'^{(2)}(x)$ is also evaluated in accordance with the limiting process

$$u'^{(2)}(x)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\{\eta_{\varkappa^2-\epsilon}(x)-\delta\varkappa^2 u_{\varkappa^2+\epsilon}(x)\}, \tag{A. 16}$$

where $u_{\varkappa^2+\epsilon}(x)$ denotes the function $u(x)$ with its mass parameter $\varkappa^2$ replaced by $\varkappa^2+\epsilon$, and $\eta_{\varkappa^2+\epsilon}(x)$ that function which is obtained by inserting $u_{\varkappa^2+\epsilon}(x)$ into $\eta(x)$ for $u(x)$.

As a suitable representation of $u_{\varkappa^2+\epsilon}(x)$, one may choose either

$$u_{\varkappa^2+\epsilon}(x)=u(x')+\frac{\epsilon}{2\varkappa^2}(x_\lambda'-x_\lambda)\partial_\lambda'\ u(x') \tag{A. 17}$$

corresponding to that of $\psi_{m-\delta m}(x)$ found by Schwinger for the case of electron, or alternatively the Fourier integral representation

$$u_{\varkappa^2+\epsilon}(x)=\int(dv)e^{iqx}\ u_{\varkappa^2+\epsilon}(q),$$
$$(q^2+\varkappa^2+\epsilon)u_{\varkappa^2+\epsilon}(q)=0. \tag{A. 18}$$

Either leads to the same result

$$u'^{(2)}(x)=Y^{(2)}u(x), \tag{A. 19}$$

which is just twice the function given by (2. 29).

According to (A. 15) and (A. 19), $u'^{(2)}(x)$ is indeterminated between 0 and $Y^{(2)}u(x)$, as is the case with the result obtained with the use of the Dyson's

method. One must, of course, take the mean value

$$u'^{(2)}(x)=\frac{1}{2}Y^{(2)}u(x). \tag{A. 20}$$

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On the Interaction of Mesons with the Electromagnetic Field. I.

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§ 1. Introduction.

At present, the meson theory is in such an unsatisfactory state that it has not proceeded much beyond its original hypothetical character and can hardly explain the experiments quantitatively. Perhaps this is partly due to the fact that various meson models we now consider do not fit the "real" meson well, and partly to the defect in the methods of calculation. The former is related to the fact that experimental evidences about mesons are very few and indirect though for electrons they are quite sufficient to construct the electron theory on them. As to the latter we have now at hand a new means of analysis, the Tomonaga-Schwinger theory[1], which has proved very useful in the case of electrons. Whether this method is also effective in its application to meson theories or not is, however, not well-known at present, especially in the case of nuclear interaction. In order to solve such problems in good approximation, it might be necessary to resort to other ways of approach, e. g., covariant analogy to the strong coupling treatment or one appropriate to the intermediate coupling. Meanwhile, it seems to be natural to consider that the Tomonaga-Schwinger theory in its present form can be applied also to mesons interacting with the electromagnetic field if some necessary modifications are made, because of the smallness of the coupling constant in this case. It is therefore expected that problems concerning mesons and electromagnetic field in interaction are free to a considerable extent from defects of calculation and that we can find out from this side an effective way of analyzing the meson models.

The interaction of the meson fields with the electromagnetic field has been investigated by various authors[2] especially in connection with the problems of polarization of mesonic vacuum, photon self-energy and the electromagnetic self-energy of a meson, most of these investigations having been confined to the examination of applicability of the Tomonaga-Schwinger theory to mesons. It is found that there appears no essential difference between the treatment of meson and electron in the e^2-approximation. We give here some results about the treatment of higher order processes, especially the dynamical reaction of the electromagnetic field to the mesonic charge-current which involves quantities to be compared with experiments in principle.

As is well-known, the Duffin-Kemmer form of equation for (scalar and

vector) mesons is preferable when it is convenient to treat mesons as particles[3]. For our purpose, it is necessary to generalize this formalism in a perfectly relativistic way which can be accomplished analogously to the Dirac equation[2]. This form of the meson theory is better than the usual Proca and Klein-Gordon form in the sense that in the Duffin-Kemmer case almost all formulas have the same form as those of the electron when γ_μ's, etc. in the latter are replaced by β_μ's etc. and thus the physical meanings of all these quantities can be better understood. Results obtained by these two alternative formalisms are, of course, identical.

Our discussion will be given in two papers, since it is too long to be published together. In this paper we shall present the covariant formulation of the meson theory and the S-matrix method describing the scattering of mesons in general (§ 2) and then apply it to find the second order dynamical reaction of electromagnetic field to the mesonic charge-current (§ 3). Effects involved in it are discussed in detail one by one, beginning with the vacuum polarization part (§ 4). Corrections of mass renormalization type (§ 5) and one of the Lamb-shift type (§ 6) as well as the discussions about the problems involved in the meson theory will be dealt with in the second paper.

§ 2. Derivation of the S-Matrix.

Let us start from a short survey over the covariant formulation of meson theory based on the Duffin-Kemmer equation. In this form of the meson theory, the free meson is represented by an operator $\psi(x)$ satisfying the following linear wave equation

$$(\beta\partial+m)\psi(x)=0 \tag{1}$$

where m is a mechanical meson mass and $\beta\partial=\sum_{\lambda=1}^{4}\beta_\lambda\frac{\partial}{\partial x_\lambda}$ ($\lambda=1, 2, 3, 4$; $x_4=it$). We shall omit the summation symbol $\sum$ for dummy suffices as usual so long as the contrary is not expressed and these suffices themselves when no complication arises. Natural unit system $\hbar=c=1$ is used throughout this paper. β_λ is a matrix satisfying the following algebraic relation

$$\beta_\lambda\beta_\mu\beta_\nu+\beta_\nu\beta_\mu\beta_\lambda=\beta_\lambda\delta_{\mu\nu}+\beta_\nu\delta_{\mu\lambda}. \tag{2}$$

As is well-known, there are just two non-trivial irreducible representations of the β-matrices, the one (5-rowed matrix representation) expressing the scalar meson, the other (10-rowed one) the vector meson. The adjoint $\psi^\dagger(x)$ of $\psi(x)$, defined by

$$\psi^\dagger(x)=\psi^*(x)(2\beta_4^2-1) \tag{3}$$

($\psi^*(x)$ is a hermitic conjugate of $\psi(x)$) satisfies

$$\partial\psi^\dagger(x)\beta-m\psi^\dagger(x)=0. \tag{1'}$$

(For this purpose it is necessary to assume that β's are all hermitian. This is

of course permissible.) Using $\psi^\dagger(x)$ and $\psi(x)$, the Lagrangian function, charge-current vector, canonical energy momentum tensor, etc. are easily constructed

$$L^m(x) = -\tfrac{1}{2}(\psi^\dagger(x)\beta\partial\psi(x) - \partial\psi^\dagger(x)\beta\psi(x)) - m\psi^\dagger(x)\psi(x),$$

$$j_\mu(x) = ie\psi^\dagger(x)\beta_\mu\psi(x),$$

$$T^m_{\mu\nu}(x) = \tfrac{1}{2}(\psi^\dagger(x)\beta_\mu\partial_\nu\psi(x) - \partial_\nu\psi^\dagger(x)\beta_\mu\psi(x)). \tag{4}$$

$j_\mu(x)$ and $T^m_{\mu\nu}(x)$ satisfy the continuity equations. The whole theory can be formulated in a charge symmetrical way which we assume implicitly performed for brevity of calculation[9].

The system of the meson field and the electromagnetic field in interaction is classically described by the following Lagrangian density

$$\mathfrak{L}(x) = L^m(x) + L^p(x) + L(x), \tag{5}$$

where $L^m(x)$ and $L^p(x)$ are Lagrangian densities describing the meson field and the electromagnetic field respectively and $L(x)$ represents the interaction between these two fields

$$L(x) = j_\mu(x)A_\mu(x). \tag{6}$$

Of course we can develop the Tomonaga-Schwinger theory in the usual way employing the canonical formalism for the Lagrangian density (5). In such a treatment the time variable has to be unnecessarily distinguished from space-variables, which not only is unsatisfactory from the relativistic viewpoint but makes the deducation of the Tomonaga-Schwinger equation rather complicated. However, it was proved by Kanesawa and Koba[4] that the same result could be obtained without referring to the canonical formulation. According to them the quantum mechanical system corresponding to any classical system (not always describable in a canonical form) can be described in the interaction representation by an equation

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma] = -L_x[\sigma]\Psi[\sigma], \tag{7}$$

where $L_x[\sigma]$ is so constructed from the usual interaction Lagrangian density that it satisfies the integrability condition

$$\frac{\delta}{\delta\sigma(x')}L_x[\sigma] - \frac{\delta}{\delta\sigma(x)}L_{x'}[\sigma] = -i[L_x[\sigma], L_{x'}[\sigma]], \tag{8}$$

for any two points x, x' which lie in a space-like direction with respect to each other. Furthermore, field variables contained in $L_x[\sigma]$ are all solutions of the free field equations satisfying the corresponding commutation relations. In our case it is found from (6) and (8) that

$$L_x[\sigma] = j_\mu(x)A_\mu(x) + J_{\mu\nu}(x;n)A_\mu(x)A_\nu(x), \tag{9}$$

with

$$J_{\mu\nu}(x\,;n) = -\frac{e^2}{m}\psi^\dagger(x)\beta_\mu(1+(\beta n)^2)\beta_\nu\psi(x), \tag{9'}$$

where n_ν is a unit vector normal to the space-like surface, $n_\nu^2=-1$. Commutation relations for the field variables appearing in (9) are

$$[\psi_\rho(x), \psi_\sigma^\dagger(x')] = \frac{1}{i}B_{\rho\sigma}D(x-x'),$$

$$[A_\mu(x), A_\nu(x')] = i\delta_{\mu\nu}D_0(x-x'), \text{ others are zero,} \tag{10}$$

where $D(x)$ and $D_0(x)$ are D-functions of meson and photon respectively and

$$B \equiv \beta\partial - \frac{1}{m}(\beta\partial)^2. \tag{10'}$$

The content of the formulation described above is, of course, identical with that of the usual canonical formalism. In fact (9) is the interaction Hamiltonian density obtained by the canonical formalism with opposite sign[4]. Therefore, the Tomonaga-Schwinger equation for the meson-photon interaction is

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma] = H_x[\sigma]\Psi[\sigma] \tag{11}$$

with

$$H_x[\sigma] = -j_\mu(x)A_\mu(x) - J_{\mu\nu}(x\,;n)A_\mu(x)A_\nu(x). \tag{12}$$

This is the starting point of our following discussions.

Introducing a unitary operator $U[\sigma]$ determined by

$$i\frac{\delta}{\delta\sigma(x)}U[\sigma] = H_x[\sigma]U[\sigma] \tag{13}$$

with $U[-\infty]=1$, we can write the solution of (11) as follows

$$\Psi[\sigma] = U[\sigma]\Psi_0 \tag{14}$$

where Ψ_0 is a constant vector representing any initial condition of the system. $U[\sigma]$ can easily be written in the form of ascending powers of $H_x[\sigma]$ by applying an iteration procedure to (13)

$$U[\sigma] = 1 + (-i)\int_{-\infty}^{\sigma}(dx')H_{x'}[\sigma'] + (-i)^2\int_{-\infty}^{\sigma}(dx')\int_{-\infty}^{\sigma(x')}(dx'')H_{x'}[\sigma']H_{x''}[\sigma''] + \cdots \tag{15}$$

which can be rewritten in the following form

$$U[\sigma] = 1 + \frac{(-i)}{1!}\int_{-\infty}^{\sigma}(dx')P(H_{x'}[\sigma']) + \frac{(-i)^2}{2!}\int_{-\infty}^{\sigma}(dx')\int_{-\infty}^{\sigma}(dx'')P(H_{x'}[\sigma'], H_{x''}[\sigma'']) + \cdots \tag{15'}$$

using the chronological ordering operator of Dyson[5]. Letting σ in (15′) tend to

infinity we obtain the so-called S-matrix

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}(dx_1)(dx_2)\cdots(dx_n)P(H_{x_1}[\sigma_1],H_{x_2}[\sigma_2],\cdots,H_{x_n}[\sigma_n]). \tag{16}$$

This is as a whole independent of the choice of families of intermediate surfaces in spite of the explicit dependence of each $H_{x_i}[\sigma_i]$ on normals to surfaces, which is the content of the integrability of (11)[6].

We shall give here a simple proof of (15′). In order to derive (15′) from (15), we have only to show that

$$\int_{-\infty}^{\sigma}(dx_1)\int_{-\infty}^{\sigma(x_1)}(dx_2)\cdots\int_{-\infty}^{\sigma(x_{n-1})}(dx_n)H_{x_1}[\sigma_1]H_{x_2}[\sigma_2]\cdots H_{x_n}[\sigma_n]$$

$$=\frac{1}{n!}\int_{-\infty}^{\sigma}(dx_1)\cdots\int_{-\infty}^{\sigma}(dx_n)P(H_{x_1}[\sigma_1],\cdots,H_{x_n}[\sigma_n]). \tag{17}$$

(17) is evident for $n=1$, since $P(H_x[\sigma])=H_x[\sigma]$. Next we suppose that (17) is true for some fixed n and define

$$G[\sigma]\equiv\int_{-\infty}^{\sigma}(dx_1)\int_{-\infty}^{\sigma(x_1)}(dx_2)\cdots\int_{-\infty}^{\sigma(x_n)}(dx_{n+1})H_{x_1}[\sigma_1]\cdots H_{x_{n+1}}[\sigma_{n+1}]$$

$$-\frac{1}{(n+1)!}\int_{-\infty}^{\sigma}(dx_1)\int_{-\infty}^{\sigma}(dx_2)\cdots\int_{-\infty}^{\sigma}(dx_{n+1})P(H_{x_1}[\sigma_1],\cdots,H_{x_{n+1}}[\sigma_{n+1}]), \tag{18}$$

then, since P is, from the definition, symmetric against the permutation of its arguments, we obtain

$$\frac{\delta G[\sigma]}{\delta\sigma(x)}=\int_{-\infty}^{\sigma}(dx_2)\int_{-\infty}^{\sigma(x_2)}(dx_3)\cdots\int_{-\infty}^{\sigma(x_n)}(dx_{n+1})H_x[\sigma]H_{x_2}[\sigma_2]\cdots H_{x_{n+1}}[\sigma_{n+1}]$$

$$-\frac{1}{n!}\int_{-\infty}^{\sigma}(dx_2)\cdots\int_{-\infty}^{\sigma}(dx_{n+1})P(H_x[\sigma],H_{x_2}[\sigma_2],\cdots,H_{x_{n+1}}[\sigma_{n+1}]). \tag{19}$$

This is equal to zero on account of (17) and

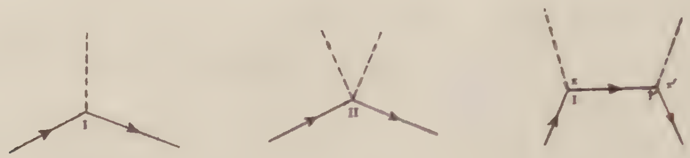
$$P(H_x[\sigma],H_{x_2}[\sigma_2]\cdots,)=H_x[\sigma]P(H_{x_2}[\sigma_2],\cdots),$$

the latter being a consequence of x lying on a surface which is future to all point $x_i(i=2,3,\cdots,n+1)$. Thus, we have found that $G[\sigma]$ is independent of σ and further identically zero on account of $G[-\infty]=0$. Therefore (17) is true for all n by mathematical induction. q. e. d.

Scattering process (in an extended sense) of any type is of course described completely by the S-matrix (16). But actual computations are largely simplified by the discussions given below. We shall write for convenience the S-matrix (16) as follows;

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}(dx_1)\cdots(dx_n)P(H^{I}(x_1)+H^{II}(x_1;n_1),\cdots,H^{I}(x_n)+H^{II}(x_n;n_n)) \tag{20}$$

where H' and H'' are the first and the second terms of the interaction Hamiltonian (12) and are represented by the graphs of vertices of type I (Fig. 1_a) and II (Fig. 1_b).



a b c

Fig. 1.

(Meson and photon lines are denoted by full and dotted lines respectively.)

As is seen by a reasoning analogous to the electron case, any process whose matrix is contained in (20) can be described graphically by a suitable combination of these two kinds of graphs. Thus we obtain various diagrams which consist of some vertices of type I and II, some internal meson and photon lines and some external meson and photon lines. The vacuum expectation value of $P(\psi^\dagger(x'), \psi(x))$, i. e., the propagation function of the meson wave corresponding to an internal meson line (Fig. 1_c) is easily found equal to

$$-\frac{1}{2}BD_F(x-x')+\frac{i}{m}(\beta n)^2\delta(x-x') \tag{21}$$

where $D_F(x)$ is the type of D-function introduced by Feynman and has the following Fourier integral representation

$$D_F(x)=\frac{-2i}{(2\pi)^4}\int(dk)\,e^{ikx}\frac{1}{k^2+m^2-i\epsilon} \tag{22}$$

in which the limit $\epsilon\to+0$ is understood. The second term of (21) appears due to the circumstance that B contains a second order derivative with respect to the normal direction to the surface which is not the case for electron.

For our purpose it is convenient to express $\langle P(\psi^\dagger(x'), \psi(x))\rangle_0$ in a slightly different form

$$-\frac{1}{2}\left\{B+\frac{1}{m}(\Box^2-m^2)\right\}D_F(x-x')+\frac{i}{m}(1+(\beta n)^2)\delta(x-x') \tag{23}$$

which is easily found to be identical with (21) when one employs the following formula

$$(\Box^2-m^2)\,D_F(x)=2i\delta(x). \tag{24}$$

The second term of (23) contributes to the S-matrix only when the end points of the meson line coincide and therefore it is appropriate to describe it by a graph similar to the one in Fig. 1_b rather than that of Fig. 1_c itself. We shall consider from now on that Fig. 1_c corresponds only to the first term of (23)

which is obviously independent of the normal to the surface[7]. The surface-dependent terms, all of them involving a factor of the form $(\beta n)^2$, appear in two places in the integrands of the S-matrix (20), one in the H'' which is surface dependent from the beginning, the other in the second term of (23) which arises for the first time when the vacuum expectation value of $P(\psi^\dagger(x'), \psi(x))$ is evaluated. They surely disappear from the final result since our S-matrix is so constructed that it does not depend on the choice of the surface families. Moreover, since $(\beta n)^2$ always appears in a combined form $1+(\beta n)^2$, we can maintain that the second term of (23) gives a contribution to the S-matrix which just cancels all effects caused by the second term H'' of the interaction Hamiltonian density (12). This is the very reason why we have preferred (23) to (21).

As a result of these considerations we can easily understand that the same S-matrix as (16) is obtained when we introduce only the first term of (12), $-j_\mu(x)A_\mu(x)$, into the P-symbol of (16) and omit all redundant terms that involve several factors of the form $1+(\beta n)^2$ which comes from the second term of (23). We shall describe this procedure symbolically by a modified P-symbol P'; then the S-matrix is expressed in the following form[8]

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}(dx_1)\cdots(dx_n)P'(-j_\mu(x_1)A_\mu(x_1),\cdots,-j_\mu(x_n)A_\mu(x_n)), \quad (25)$$

which is much easier to treat than the original form (16).

The above considerations are not sufficient since no account is taken of the effect of electromagnetic interaction on the mass of the meson field. By direct calculation in e^2-approximation (see § 5), the electromagnetic self-energy of a free meson is shown to be a linear combination of two terms $\psi^\dagger\psi$ and $\psi^\dagger\beta_\rho^2\psi$. (This is due to the circumstance that β_ρ^2 is not a multiple of unit matrix contrary to the case of Dirac electron for which $\gamma_\rho^2=4$.) This fact seems to bring some complication into our discussions. They induce, however, similar effects in the S-matrix, as they are in the case of free mesons connected to each other by the equation

$$5\psi^\dagger(x)\psi(x)-2\psi^\dagger(x)\beta_\rho^2\psi(x)=\frac{1}{2m}\partial_\nu[\psi^\dagger(x)(\beta_\rho^2\beta_\nu-\beta_\nu\beta_\rho^2)\psi(x)] \quad (26)$$

which is easily verified using (1) and (1').

We have thus sufficient reasons to suppose that the self-energy plays no role other than changing the mass of a meson and cannot be distinguished at all from the mechanical mass. To justify this assertion, it is necessary to show that it is possible to remove the self-energy of the meson field H^{self} from (11) and thereby alter the equation of motion for the meson field (1) into that of a meson with modified mass. This seems to be effected by the state vector transformation

$$\Psi[\sigma]=W[\sigma]\,\Phi[\sigma], \quad (27)$$

where $W[\sigma]$ is designed to remove the variation associated with H^{self} from $\Psi[\sigma]$ and therefore is subject to the equation of motion

$$i\frac{\delta}{\delta\sigma(x)}W[\sigma]=H^{\rm self}(x)W[\sigma]. \tag{28}$$

However, (28) has no solution at all, since it does not satisfy the integrability condition as is easily verified by direct calculation. Moreover, the equation of motion which is to describe the meson in terms of the experimental mass

$$i\frac{\delta}{\delta\sigma(x)}\Phi[\sigma]=W^{-1}[\sigma](H_x[\sigma]-H^{\rm self}(x))W[\sigma]\Phi[\sigma] \tag{29}$$

is not integrable even if $W[\sigma]$ is determined by a suitable modification of the equation (28). This method, which was successfully employed by Schwinger to treat the self-energy of an electron, is thus found to be difficult to apply imme i-ately to the meson case. In order to avoid these difficulties, we shall proceed in a somewhat different way as follows.

Our theory has heretofore been constructed on the assumption that the interacting meson-photon system can be described by equation (7) where the field variables are supposed to satisfy the free field equations with mechanical masses. But we may reconstruct our theory in a slightly different way; let us suppose that our system is described by equation (7) and the integrability condition (8) where the meson operator is considered to be a solution of the equation of motion for a free meson propagating with experimental mass m (mechanical mass m_0 combined with the electromagnetic mass δm which is supposed to be small in a future correct theory though divergent in the present theories). As a result of this alteration we must use

$$L'(x)=j_\mu(x)A_\mu(x)+\delta m\phi^\dagger(x)\phi(x) \tag{30}$$

as an interaction Lagrangian density in place of (6). Further discussions are made almost in the same way as before, i. e., construction of $L'_x[\sigma]$ from $L'(x)$, derivation of the S-matrix, etc. We shall not give here the explicit form of $L'_x[\sigma]$, since it is not so easily obtained as (9) and the terms modifying (30) disappear entirely from the S-matrix in the final stage of calculation[9]. The second term of (30) is easily seen to cancel the self-energy effect in the S-matrix. (This will further be discussed in detail in § 5.) We have thus arrived at the result desired.

Therefore it is concluded that the Dyson's program for the electron-photon system can be applied to the case of meson-photon system without any formal change if appropriate attention is paid to the necessary modifications relating to the integrability condition of the theory. However, nothing more than formalistic is involved in this conclusion, since the real difficulties of meson theory lie not in the way of treatment described above but in the appearance of divergent quantities which cannot be removed through the ordinary renormalization procedure.

The elementary divergent integrals can easily be discovered and investigated by the method of Dyson[10]. Obviously, there appear such integrals more frequently

in our theory than in the electron case, since they have the same structure save for the fact that the commutation relation of the former involves derivatives of higher order than the latter. Thus, for example, it is found that the fourth order Møller scattering diverges even in the case of the scalar mesons. We shall not give here a closer examination about them[11].

§ 3. Second Order Corrections to the Mesonic Charge-Current.

In the preceding section a covariant formulation of the interaction of meson fields with the electromagnetic field has been developed. We shall now be concerned with its application to the actual problems. The lowest order processes such as the polarization of mesonic vacuum, the photon self-energy and the electromagnetic self-energy of a meson have heretofore been investigated by analyzing the second order Hamiltonian obtained by a contact transformation which is useful to describe the absence of real first order processes[2]. It is however easily seen that this method is too clumsy to be applied to higher order processes contrary to the simple and perspective S-matrix method adopted here. In the following sections we shall evaluate the second-order corrections to the mesonic charge-current as an example of the S-matrix method described in § 2. This involves quantities which are to be compared with the results of experiments. In fact, some of them are divergent and give rise to the serious difficulties of the meson theory since they cannot be concealed in unobservable charge and mass renormalization factors.

For the following discussions it is convenient to distinguish the electromagnetic field from the external force. Therefore we shall denote them by $A_\mu(x)$ and $A^e_\mu(x)$ respectively. Then, the S-matrix is expressed as follows

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}(dx_1)\cdots(dx_n)P'(-j_\mu(x_1)A^e_\mu(x_1)-j_\mu(x_1)A_\mu(x_1)-\delta m\phi^\dagger(x_1)\phi(x_1),$$
$$,\cdots,\ -j_\mu(x_n)A^e_\mu(x_n)-j_\mu(x_n)A_\mu(x_n)-\delta m\phi^\dagger(x_n)\phi(x_n)). \tag{31}$$

In order to find the correction to the mesonic charge-current in the S-matrix (31), the simplest way is to discuss the second order correction to the elastic scattering of a meson scattered once by an external electromagnetic potential. The diagram which describes the elastic scattering of a meson by an external force is given in Fig. 2. in which the letter e indicates the external potential, the corresponding matrix element being

$$-i\int_{-\infty}^{\infty}(dx)[-ie\phi^\dagger(x)\beta_\mu\phi(x)A^e_\mu(x)], \tag{32}$$

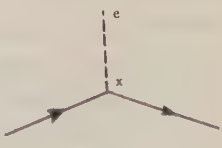


Fig. 2.

which is an integral of the energy of external field over all space-time region. The diagrams representing the second order corrections to the elastic scattering of a meson are obtained by modifying

the vertex and the lines in Fig. 2 to the second order. They are given in Fig. 3.

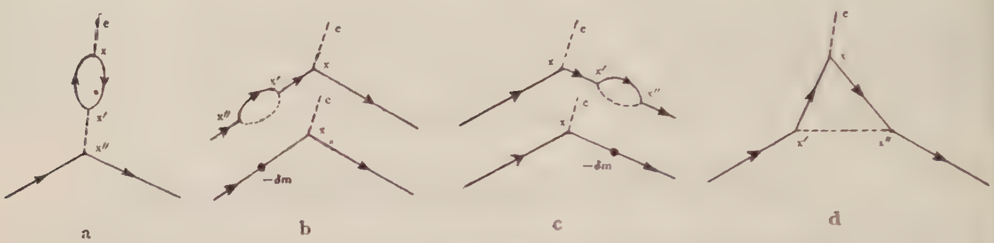


Fig. 3.

These diagrams give rise to the integrals

$$M = M_a + M_b + M_c \tag{33}$$

with

$$M_a = -\frac{\pi}{2}\,\alpha e \int (dx)(dx')(dx'') A^e_\mu(x) T_r[\beta_\mu B D_F(x-x')\beta_\nu B D_F(x'-x)]$$
$$\times D_{0F}(x'-x'')\psi^\dagger(x'')\beta_\nu\psi(x''), \tag{34a}$$

$$M_b = -\frac{\pi}{2}\,\alpha e \int (dx)(dx')(dx'') A^e_\mu(x)[\psi^\dagger(x)\beta_\mu B D_F(x-x')\beta_\nu B D\;(x'-x'')$$
$$\times \beta_\nu D_{0F}(x'-x'')\psi(x'') + \psi^\dagger(x'')\beta_\nu B D_F(x''-x')\beta_\nu D_{0F}(x''-x')$$
$$\times B D_F(x'-x)\beta_\mu\psi(x)] + \frac{ie}{2}\,\delta m \int (dx)(dx') A^e_\mu(x)[\psi^\dagger(x)\beta_\mu B D_F(x-x')$$
$$\times \psi(x') + \psi^\dagger(x') B D_F(x'-x)\beta_\mu\psi(x)], \tag{34b}$$

$$M_c = -\frac{\pi}{2}\,\alpha e \int (dx)(dx')(dx'') A^e_\mu(x)\psi^\dagger(x')\beta_\nu B D_F(x-x')\beta_\mu B D_F(x''-x)$$
$$\times \beta_\nu\psi(x'') D_{0F}(x'-x''), \tag{34c}$$

where $\alpha = \dfrac{e^2}{4\pi}$ is the Sommerfeld's fine structure constant and $D_{0F}(x)$ is the Feynman's D-function for the electromagnetic field

$$D_{0F}(x) = \lim_{\epsilon\to+0} \frac{-2i}{(2\pi)^4}\int (dk) e^{ikx} \frac{1}{k^2 - i\epsilon}. \tag{35}$$

For convenience, $B + \dfrac{1}{m}(\Box^2 - m^2)$ is denoted simply by B in (34) and the succeeding formulas according to the discussion concerning the formula (23). All of these integrals having the form of integrals of the energy of external field, the factors multiplying $A^e_\mu(x)$ have to be regarded as the corrections to the mesonic charge-current. Thus we obtain

$$\delta j_\mu(x) = \delta j^a_\mu(x) + \delta j^b_\mu(x) + \delta j^c_\mu(x) \tag{36}$$

ith

$$\delta j_\mu^a(x)=i\frac{\pi}{2}uc\int(dx')(dx'')T_r[\beta_\mu BD_F(x-x')\beta_\nu BD_F(x'-x)]$$
$$D_{0F}(x'-x'')\psi^\dagger(x'')\beta_\nu\psi(x''), \tag{37a}$$

$$\delta j_\mu^b(x)=i\frac{\pi}{2}uc\int(dx')(dx'')[\psi^\dagger(x)\beta_\mu BD_F(x-x')\beta_\nu BD_F(x'-x'')$$
$$\times\beta_\nu D_{0F}(x'-x'')\psi(x'')+\psi^\dagger(x'')\beta_\nu BD_F(x''-x')\beta_\nu D_{0F}(x''-x')$$
$$\times BD_F(x'-x)\beta_\mu\psi(x)]+\frac{c}{2}\delta m\int(dx')[\psi^\dagger(x)\beta_\mu BD_F(x-x')\psi(x')$$
$$+\psi^\dagger(x')BD_F(x'-x)\beta_\mu\psi(x)], \tag{37b}$$

$$\delta j_\mu^c(x)=i\frac{\pi}{2}uc\int(dx')(dx'')\psi^\dagger(x')\beta_\nu BD_F(x-x')\beta_\mu BD_F(x''-x)$$
$$\times\beta_\nu\psi(x'')D_{0F}(x'-x''). \tag{37c}$$

In the following sections we shall discuss in some detail these three corrections in their order.

§ 4. The Polarization of the Mesonic Vacuum.

The integral (37a) can be rewritten in the form

$$\delta j_\mu^a(x)=-i\pi u\int(dx')T_r[\beta_\mu BD_F(x-x')\beta_\nu BD_F(x'-x)]\delta A_\nu(x') \tag{38}$$

where

$$\delta A_\nu(x')=-\frac{c}{2}\int(dx'')D_{0F}(x'-x'')\psi^\dagger(x'')\beta_\nu\psi(x''). \tag{39}$$

Expressed in this form it is obvious that (37a) is the current induced by the electromagnetic field that accompanies a mesonic charge-current distribution. This can be treated in the same way as the polarization of the vacuum by an external field which has been discussed in the c^2-approximation by various authors[2]. We shall, for convenience, write (38) as

$$\delta j_\mu^a(x)=-i\pi u\int(dx')G_{\mu\nu}(x-x')\delta A_\nu(x') \tag{40}$$

where

$$G_{\mu\nu}(x)=T_r[\beta_\mu B(x)D_F(x)\beta_\nu B(-x)D_F(x)] \tag{41}$$

and $B(x)$ means that the differentiation involved in B is to be performed with respect to x.

In order to evaluate $G_{\mu\nu}(x)$, we shall substitute the Fourier integral repre-

sentation (22) for the $D_F(x)$ in (41). (Our method of calculation is almost the same with Schwinger's.) Then

$$G_{\mu\nu}(x)=\left(\frac{-2i}{(2\pi)^4}\right)^2\int(dk')(dk'')e^{i(k'+k'')x}(k'^2+m^2)^{-1}(k''^2+m^2)^{-1}$$

$$\times T_r\left[\beta_\mu\left\{i(k'\beta)+\frac{1}{m}(k'\beta)^2-m-\frac{1}{m}k'^2\right\}\beta_\nu\left\{-i(k''\beta)+\frac{1}{m}(k''\beta)^2-m-\frac{1}{m}k''^2\right\}\right]. \tag{42}$$

To evaluate the trace in (42), we employ the following formulas for the β-matrices[7]

$$\begin{aligned}
T_r\beta_\lambda\beta_\mu &= \delta_{\lambda\mu}\sigma_1\\
T_r\beta_\lambda\beta_\mu\beta_\nu &= 0\\
T_r\beta_\lambda\beta_\mu\beta_\nu\beta_\rho &= \tfrac{1}{2}\sigma_2(\delta_{\lambda\mu}\delta_{\nu\rho}+\delta_{\lambda\nu}\delta_{\mu\rho})\\
T_r\beta_\lambda\beta_\mu\beta_\nu\beta_\rho\beta_\sigma &= 0\\
T_r\beta_\lambda\beta_\mu\beta_\nu\beta_\rho\beta_\sigma\beta_\tau &= \sigma_3(\delta_{\lambda\mu}\delta_{\nu\rho}\delta_{\sigma\tau}+\delta_{\mu\nu}\delta_{\rho\sigma}\delta_{\tau\lambda})\\
&\quad+(\tfrac{1}{2}\sigma_1-\sigma_3)(\delta_{\lambda\mu}\delta_{\nu\tau}\delta_{\rho\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma}\delta_{\lambda\tau}+\delta_{\sigma\tau}\delta_{\rho\lambda}\delta_{\mu\nu}-\delta_{\lambda\rho}\delta_{\sigma\mu}\delta_{\tau\nu})
\end{aligned} \tag{43}$$

where

$$\sigma_1\equiv T_r\beta_1^2=\tfrac{1}{4}T_r\beta_\rho^2=\begin{cases}2 & \text{for spin 0}\\ 6 & \text{for spin 1}\end{cases}$$

and

$$\sigma_3\equiv T_r\beta_1^2\beta_2^2\beta_3^2=1 \qquad \text{for both spins.}$$

Thus we obtain

$$T_r\left[\beta_\mu\left\{i(k'\beta)+\frac{1}{m}(k'\beta)^2-m-\frac{1}{m}k'^2\right\}\beta_\nu\left\{-i(k''\beta)+\frac{1}{m}(k''\beta)^2-m-\frac{1}{m}k''^2\right\}\right]$$

$$=\frac{1}{2}\sigma_1\{-(k'_\mu-k''_\mu)(k'_\nu-k''_\nu)+\delta_{\mu\nu}(k'^2+m^2+k''^2+m^2)\}$$

$$+\frac{1}{m^2}\left(\sigma_3-\frac{1}{2}\sigma_1\right)\{k'_\mu k'_\nu k''^2+k''_\mu k''_\nu k'^2-(k'k'')(k'_\mu k''_\nu+k''_\mu k'_\nu)+\delta_{\mu\nu}((k'k'')^2-k'^2k''^2)\} \tag{44}$$

(42) satisfies the requirements of gauge invariance and charge conservation if the value of divergent integral is suitably determined. In fact, on computing $\partial G_{\mu\nu}(x)/\partial x_\mu$ from (42) and (44), we obtain

$$\frac{\partial}{\partial x_\mu}G_{\mu\nu}(x)=-\frac{8}{(2\pi)^8}\int(dk')(dk'')e^{i(k'+k'')x}\frac{k'_\nu(k''^2+m^2)+k''_\nu(k'^2+m^2)}{(k'^2+m^2)(k''^2+m^2)}$$

$$=-\frac{8}{(2\pi)^8}\int(dp)e^{ipx}\left[\int(dk')\frac{k'_\nu}{k'^2+m^2}+\int(dk'')\frac{k''_\nu}{k''^2+m^2}\right] \tag{45}$$

where

$$p_\mu=k'_\mu+k''_\mu, \tag{46}$$

which is indeed zero if

$$\int (dk')\frac{k'_\nu}{k'^2+m^2}=0 \quad \text{and} \quad \int (dk'')\frac{k''_\nu}{k''^2+m^2}=0. \tag{47}$$

The vanishing of the latter integral seems to be natural since it is an integral of an odd function over the entire energy-momentum space though it is divergent.

The denominator of the integrand of (42) can be simplified in the following manner:

$$(k'^2+m^2)^{-1}(k''^2+m^2)^{-1}=\frac{1}{2}\int_{-1}^{+1}dv\left(\frac{k'^2+k''^2}{2}+\frac{k'^2-k''^2}{2}v+m^2\right)^{-2}. \tag{48}$$

Let us introduce the new variables k_μ and p_μ as defined by

$$k'_\mu=\frac{1}{2}p_\mu+\left(k_\mu-\frac{v}{2}p_\mu\right),$$

$$k''_\mu=\frac{1}{2}p_\mu-\left(k_\mu-\frac{v}{2}p_\mu\right), \tag{49}$$

then $G_{\mu\nu}(x)$ is brought into the form

$$G_{\mu\nu}(x)=-\frac{1}{(2\pi)^8}\int(dk)(dp)e^{ipx}\int_{-1}^{+1}dv\left(k^2+m^2+\frac{p^2}{4}(1-v^2)\right)^{-2}$$
$$\times\left[\sigma_1 v^2(\delta_{\mu\nu}p^2-p_\mu p_\nu)+\left(\frac{1}{2}\sigma_1-\sigma_3\right)\frac{k^2}{m^2}(\delta_{\mu\nu}p^2-p_\mu p_\nu)+\delta_{\mu\nu}\sigma_1\left(k^2+2m^2+\frac{p^2}{2}(1-v^2)\right)\right], \tag{50}$$

where, in virtue of the dependence of the denominator on k^2 alone, terms involving k_μ and $k_\lambda k_\mu k_\nu$ have been discarded, and $k_\mu k_\nu$ has been replaced by $\frac{1}{4}\delta_{\mu\nu}k^2$. It is the last term of the integrand of (50) that makes the definition (47) necessary to guarantee the requirement of the gauge invariance. It seems to arise due to the defect in the treatment of diverging integrals whose real solution has not yet been obtained though several attempts have been made to solve this difficulty[13]. At any rate, we may be allowed to suppose that the term in consideration disappears from (50) in the future correct theory in which the gauge invariance is automatically assured at each step of calculation. Therefore, we shall drop this term from (50), then it can be written in the form

$$G_{\mu\nu}(x)=(\partial_\mu\partial_\nu-\delta_{\mu\nu}\square^2)G(x), \tag{51}$$

where

$$G(x)=\sigma_1 G_1(x)+(\tfrac{1}{2}\sigma_1-\sigma_3)G_2(x) \tag{52}$$

with

$$G_1(x)=-\frac{2}{(2\pi)^8}\int_0^1 v^2 dv\int(dk)(dp)e^{ipx}\frac{1}{\left(k^2+m^2+\frac{p^2}{4}(1-v^2)\right)^2}, \tag{53$_1$}$$

$$G_2(x)=-\frac{2}{(2\pi)^8}\int_0^1 dv\int(dk)(dp)e^{ipx}\frac{\dfrac{k^2}{m^2}}{\left(k^2+m^2+\dfrac{p^2}{4}(1-v^2)\right)^2}. \tag{53_2}$$

By a partial integration with respect to v, they become

$$G_1(x)=-\frac{2}{3(2\pi)^8}\int(dk)(dp)e^{ipx}\left[\frac{1}{(k^2+m^2)^2}-p^2\int_0^1\frac{v^4dv}{\left(k^2+m^2+\dfrac{p^2}{4}(1-v^2)\right)^3}\right], \tag{54_1}$$

$$G_2(x)=-\frac{2}{(2\pi)^8}\int(dk)(dp)e^{ipx}$$
$$\times\left[\frac{1}{m^2(k^2+m^2)}-\frac{1+\dfrac{1}{3}\dfrac{p^2}{m^2}}{(k^2+m^2)^2}+\frac{p^2}{m^2}\int_0^1\frac{v^2dv\left(m^2+\dfrac{p^2}{4}\left(1+\dfrac{1}{3}v^2\right)\right)}{\left(k^2+m^2+\dfrac{p^2}{4}(1-v^2)\right)^3}\right] \tag{54_2}$$

The integration with respect to k can now easily be performed. The formulas required for this purpose are (when expressed in the three dimensional notation)

$$\int(dk)\frac{1}{k^2+m^2}=\lim_{K\to\infty}2i\pi^2\left(KK_0-m^2\log\frac{K+K_0}{m}\right), \tag{55}$$

$$\int(dk)\frac{1}{(k^2+m^2)^2}=\lim_{K\to\infty}2i\pi^2\left(\log\frac{K+K_0}{m}-1\right), \tag{56}$$

$$\int(dk)\frac{1}{(k^2+\Lambda^2)^3}=\frac{i\pi^2}{2\Lambda^2} \tag{57}$$

where

$$K_0=(K^2+m^2)^{\frac{1}{2}}. \tag{58}$$

(In the following calculations $\lim\limits_{K\to\infty}$ is omitted for simplicity.) Using these formulas, (54_1) and (54_2) become

$$G_1(x)=-\frac{i\delta(x)}{3(2\pi)^2}\left(\log\frac{K+K_0}{m}-1\right)-\frac{i}{12(2\pi)^2}\frac{\Box^2}{m^2}F_2(x), \tag{59_1}$$

$$G_2(x)=\frac{i}{(2\pi)^2}\left[-\frac{1}{m^2}\left(KK_0-m^2\log\frac{K+K_0}{m}\right)+\left(\log\frac{K+K_0}{m}-1\right)\left(1-\frac{1}{3}\frac{\Box^2}{m^2}\right)\right.$$
$$\left.+\frac{1}{12}\frac{\Box^2}{m^2}\right]\delta(x)-\frac{i}{12(2\pi)^2}\left(\frac{\Box^2}{m^2}\right)^2F_2(x) \tag{59_2}$$

where

$$F_n(x)=\frac{1}{(2\pi)^4}\int_0^1 v^{2n}dv\int(dp)\frac{e^{ipx}}{1+\dfrac{p^2}{4m^2}(1-v^2)} \tag{60}$$

Finally, we may insert (51) into (40) and integrate by parts to obtain

$$\delta j_\mu^a(x) = i\pi a \int (dx') G(x-x') \Box'^2 \delta A_\mu(x'), \tag{61}$$

where the term $\partial_\mu \partial_\nu$ is dropped, since the electromagnetic potential obeys the Lorentz condition. Now, since

$$\begin{aligned} \Box'^2 A_\mu(x') &= \Box'^2 \left(\frac{i}{2}\right) \int (dx'') D_{0F}(x'-x'') j_\mu(x'') \\ &= -j_\mu(x') \end{aligned} \tag{62}$$

from (39), (61) may be rewritten as

$$\delta j_\mu^a(x) = -i\pi a \int (dx') G(x-x') j_\mu(x'). \tag{63}$$

The expression of $G(x)$ contained in (59) then yields

$$\begin{aligned} \delta j_\mu^a(x) = &-\frac{a}{12\pi}\sigma_1 \left[\left(\log\frac{K+K_0}{m}-1\right) j_\mu(x) + \frac{1}{4}\frac{\Box^2}{m^2}\int (dx') F_2(x-x') j_\mu(x')\right] \\ &+\frac{a}{4\pi}\left(\frac{1}{2}\sigma_1-\sigma_2\right)\left[\left\{-\frac{1}{m^2}\left(KK_0-m^2\log\frac{K+K_0}{m}\right)+\left(\log\frac{K+K_0}{m}-1\right)\left(1-\frac{1}{3}\frac{\Box^2}{m^2}\right)\right.\right. \\ &\left.\left.+\frac{1}{12}\frac{\Box^2}{m^2}\right\} j_\mu(x) - \frac{1}{12}\left(\frac{\Box^2}{m^2}\right)^2 \int (dx') F_2(x-x') j_\mu(x')\right]. \end{aligned} \tag{64}$$

Inserting the values of σ_1 and σ_2 for spin 0 and 1, we finally arrive at

$$\delta j_\mu^a(x)_{\mathrm{sc.}} = -\frac{a}{6\pi}\left(\log\frac{K+K_0}{m}-1\right) j_\mu(x) - \frac{a}{24\pi}\frac{\Box^2}{m^2}\int (dx') F_2(x-x') j_\mu(x') \tag{65}$$

for scalar mesons, and

$$\begin{aligned} \delta j_\mu^a(x)_{\mathrm{vec.}} = &-\frac{a}{2\pi}\left(\frac{KK_0}{m^2}-\log\frac{K+K_0}{m}\right) j_\mu(x) \\ &-\frac{a}{6\pi}\left(\log\frac{K+K_0}{m}-\frac{5}{4}\right)\frac{\Box^2}{m^2} j_\mu(x) \\ &-\frac{a}{8\pi}\frac{\Box^2}{m^2}\left(1+\frac{1}{3}\frac{\Box^2}{m^2}\right)\int (dx') F_2(x-x') j_\mu(x'). \end{aligned} \tag{66}$$

for vector mesons. The first terms of (65) and (66) are proportional to the current operator. Though their coefficients are infinite, they have no observable consequences, since they are indistinguishable from the original current $j_\mu(x)$. We can therefore do away with these terms through the well-known procedure of charge renormalization.

There remains, however, one more divergent factor multiplying $\frac{\Box^2}{m^2} j_\mu(x)$ in

(66) after the first term is removed by the charge renormalization. This term is to describe the observable phenomenon and cannot be removed by any usual renormalization technique. This divergence seems to indicate one of the deep-seated difficulties of the meson theory whose solution probably necessitates some knowledge about the structure of such particles. This term can, of course, be removed by various methods, e. g., the regularization method proposed by Pauli and Villars[13]. In our opinion, however, these methods are not satisfactory. Such defects of theories as the divergence discussed above should be utilized as a step for further investigations and not be circumvented by devices whose physical meanings are rather ambiguous.

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Letters to the Editor

On the Positive Excess of the Hard Component in Cosmic-Ray

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The phenomena of the positive excess of the hard component give a strong support to Proton Primary Hypothesis and also offer us some knowledge about the production of the meson. Before going to investigate the spectrum and the multiplicity of the meson production, we analyze the problem of the subject.

Let H^+ and H^- be the intensity of the hard component of positive and negative charge respectively and we define the positive excess T as follows,

$$T=2(H^+-H^-)/(H^++H^-).$$

Various experimental results are represented in Fig. (1). We summarize below the remarkable points that are obtained from these results. (i): At sea level,[1)–3)] $T=20\pm5\%$. T diminishes in high energy ($\gtrsim$2Bev) as well as in low energy[3)] ($\lesssim$0.5Bev) part of the meson component, (ii): At the atmospheric depths 3mH_2O[4)] and 7.2mH_2O,[5)] the positive excess definitely vanishes in the low energy ($\lesssim$0.5Bev). (iii); Comparing (ii) with the results at the depths 5.5mH_2O and 4mH_2O,[6)] we perceive that the meson component ($\gtrsim$0.5Bev) at the depth 5.5mH_2O has——noting that the intensity of proton amounts only to 5% of the hard component and contribute at most 10% to the positive excess——positive excess of 20%, while it loses the positive excess in traversing through the atmosphere. Unless we assume the excentric distribution of the positive excess over the spectrum of the meson component, we should accept the existence of such meson production, even in the lower atmosphere, that diminishes the positive excess. (iv): The result with cloud chamber at 2.9mH_2O is obtained by Anderson et al.[7)] and is represented in the figure after the subtraction of the contribution due to proton. It differs considerably from Conversi's result[4)] with the circuit of delay coincidence. However, the difference is not serious if we take into account the π-meson production by the proton in the apparatus of Conversi's experiment. (v): The variation of the positive excess with the energy E of the meson component at sea level is expressed as $T\propto E^{-\frac{1}{2}}$[8),9)]

We shall discuss the multiplicity and the spectrum of the meson production by the nucleon-nucleon collision according as the knowledges above represented in the next letter.

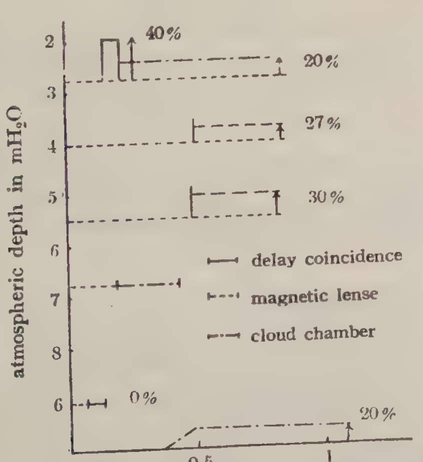

Fig. 1. momentum in Bev/c

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The Determination of the Multiplicity and the Spectrum of the Meson produced by the Nucleon-Nucleon Collision

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We investigate the relation between the multiplicity of the meson production by the collision of the primary proton with air nuclei and the positive excess of the produced mesons according as the results summarized in the previous letter (cited as I). The positive excess of the produced mesons does not depend on the type of the production process——plural or multiple——, if only the conservation of charge is taken into account. Let n be the number of the produced charged mesons, then it reults in $T=1/n$.

Now, the intensity of the proton at the atmospheric depth x_0 in the energy range ε and $\varepsilon+d\varepsilon$ is assumed to be

$A\exp(-x_0/l)\varepsilon^{-\gamma-1}d\varepsilon$, $\gamma=1.8$ $l=1.25\text{m } H_2O$.

And we put the spectrum of the production of π-meson in energy E by a proton of energy ε in the form $c\varepsilon^{\alpha}E^{\beta-1}$. Then, the multiplicity becomes

$$n=c\int_{\varkappa}\varepsilon^{\alpha}E^{\beta-1}dE\simeq-(c/\beta)(\varepsilon/\varkappa)^{\alpha}. \qquad (1)$$

If the energy loss of a proton is proportional to its primary energy, it results in $\alpha+\beta=0$. Since n increases with ϵ, α must be positive number. Remembering that three-fourth of the energy of a π-meson is transferred to a μ-meson in the π-μ decay and that the intensity of the μ-meson varies due to the collision loss and the decay with the atmospheric depth, we can express the μ-meson intensity at the depth x and with energy ϵ as

$$\mu(\epsilon,x)=\frac{Ac}{l}\left(\frac{4}{3}\right)^{\beta}\int_0^x e^{-x_0/l}dx_0\int\varepsilon^{\alpha-\gamma-1}$$
$$d\varepsilon\{\epsilon+s(x-x_0)\}^{\beta-1}\left\{\frac{x_0}{x}\frac{\epsilon}{\epsilon+s(x-x_0)}\right\}^{b/\cdot+sx}.$$

Then, the mean positive excess of the μ-meson at the depth x becomes

$$T_\mu(\epsilon,x)=\frac{Ac}{l}\left(\frac{4}{3}\right)^{\beta}\int_0^x e^{-x_0/l}dx_0\int\varepsilon^{\alpha-\gamma-1}$$
$$d\varepsilon\{\epsilon+s(x-x_0)\}^{\beta-1}\left\{\frac{x_0}{x}\frac{\epsilon}{\epsilon+s(x-x_0)}\right\}^{b/\ +sx}$$
$$(1/n(\varepsilon))/\mu(\epsilon,x)$$

Considering it in the lower atmosphere, we can integrate T_μ approximately,

$$T_\mu(\epsilon,x)=\begin{cases}-(\beta/c\gamma)(3/4)^{\alpha}\varkappa^{\alpha}/(\epsilon+\beta x)^{-\beta} & \\ \qquad (\gamma-\alpha) \quad \frac{4}{3}(\epsilon+sx)\geq E_c & \\ -(\beta/c\gamma)(\varkappa/E_c)^{\alpha}(\gamma-\alpha) & \\ \qquad \frac{4}{3}(\epsilon+sx)\leq E_c & \end{cases} \qquad (2)$$

E_c means the geomagnetic cut-off energy. From (v) in I, we take α equal to $\frac{1}{2}$. T_μ becomes

$$T_\mu(\epsilon,x)=\begin{cases}\frac{1.3}{3.6}\left(\frac{3}{4}\right)^{1/2}(\varkappa/\epsilon+sx)^{1/2} & \\ \qquad \frac{4}{3}(\epsilon+sx)\geq E_c & (3) \\ \frac{1.3}{3.6}\left(\frac{\varkappa}{E_c}\right)^{1/2} \quad \frac{4}{3}(\epsilon+sx)<E_c & (3')\end{cases}$$

The results (i) in I are obtained at the latitude near 45°N. Considering the meson

component with not so high energy ($\lesssim$2Bev), we obtain

$$\frac{1.3}{3.6c}\left(\frac{0.15}{4}\right)=1/5, \quad \therefore\ c=0.3\pm0.05$$

Thus, we can get the multiplicity of the charged meson produced by a proton with energy ε, such as $n(\varepsilon)=0.6(\varepsilon/\varkappa)^{1/2}$. In the phenomenological treatment above, n is the total number of the produced charged meson from a proton, not concerning to how many times can a proton collide with the air nuclei before it slows down. It may also be noted that (3)′ indicates the latitude effect of the positive excess. We expect the positive excess to be 10% at the equator on sea level.

Next, the facts (ii) and (iii) in I cannot be explained by such a simple process of the meson production as above. There exists γ-ray as the agent which are more likely to produce negatively charged meson than positively charged one. In fact, concerning to the produced π-meson by γ-ray, it has been known that $\pi^-:\pi^+=1.7$ in the experiment at Berkeley. Other evidences of the existence of the meson-producible γ-ray are detected in the experiments about cosmic-ray.[1] On the other hand, we can not neglect the effect due to nucleon component,[2] especially due to neutron. In the lower energy process ($\lesssim$2Bev), the contribution due to neutron may be expected to overwhelm the one due to proton.[3] Anyhow, in order to make clear what is the agent effective to produce the slow meson ($\lesssim$0.5Bev), further analysis both theoretical and experimental are required.

In conclusion, I wish to express my deep gratitude to Prof. S. Sakata for his valuable discussions about this work.

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On the New Mode of Interaction between Spinor Fields

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Recently, C. N. Yang[1] pointed out that there may be another spinor fields which are distinguished from ordinary one by the transformation character with respect to reflection. We may call these spinor fields as "pseudo spinor fields". The introduction of these fields are expected to bring about new modes into Fermi's interactions between spinor fields. The chief points of this note are to investigate (1) the influence of new modes on the meson decay processes and (2) the determination of the interaction models between elementary particles.

(A) Meson decay.

(1) μ-e decay.

Tiomno and Wheeler[2] treated the cases $\mu\to e+2\nu$ and $\mu\to e+\mu_0+\nu$ in detail. The introduction of pseudospinor field affects only the terms proportional to the rest mass of these Fermi particles and the existence of neutrino makes these terms vanish, so the new modes of interaction influence only the alternations of the name of coupling. If we adopt the model $\mu\to e+2\mu_0$, new results appear. The rest mass of μ_0-meson can be decided by the maximum energy of the decay electron. Taking this as 55 *Mev*[5] and μ-meson mass as 216 m_e, μ_0-meson mass must be negligibly small, but taking as 50 *Mev.*, μ_0-meson mass is about 30 m_e. In this respect more precise measurements are desired. If μ_0-meson mass is large, the energy distribution curves of the decay electron are changed. Contrary to the case $\mu\to e+2\nu$, the transition probabilities of $\mu\to e+2\mu_0$ are always zero at the maximum energy of decay electron. In nuclear β decay processes, new modes do not also appear, because of the existence of neutrino.

(2) $\pi \to \mu + \mu_0$.

Tiomno and Wheeler[4] showed that in the model of Fig. 1, $g_a \simeq g_b$. In this case, the processes $\pi \to \mu + \mu_0$ and $\pi \to e + \nu$ occur through the intermediary of nucleon field. The ratio of the transition matrix elements for $\pi \to e + \nu$ and $\pi \to \mu + \mu_0$ decay was shown to be smaller than $1(10^{-4})$ only for the follwing case.[5]

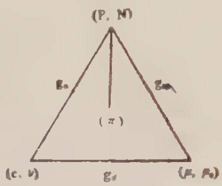

Fig. 1.

π	coupling (π-N)	NP	μ-μ_0	Fermi
ps	ps or pv	○	○	$(\bar{\varphi}\gamma_5\gamma_\alpha\psi)(\bar{\xi}\gamma_5\gamma_\alpha\eta)$

Introducing pseudospinor field, following combinations are also allowed.

π	coupling (π-N)	NP	μ-μ_0	Fermi
ps	ps or pv	○	△	$(\bar{\varphi}\gamma_5\gamma_\alpha\psi)(\bar{\xi}\gamma_\alpha\eta)$
s	s	△	○	$(\bar{\varphi}\gamma_\alpha\psi)(\bar{\xi}\gamma_5\gamma_\alpha\eta)$
s	s	△	△	$(\bar{\varphi}\gamma_\alpha\psi)(\bar{\xi}\gamma_\alpha\eta)$

where φ, ψ, ξ and η are the wave functions of neutron, proton, μ meson and μ_0 meson respectively and ○ is the case when both proton and neutron are the same spinor fields and △ is the case when proton and neutron are different spinor.

(B) Interaction models between elementary particles.

The follwing considerations are essentially based on the charge symmetrical theory for π meson. From the existence of the spin orbital force in neutron-neutron scattering[6] and the γ-decay of neutretto,[7] neutretto may be pseudoscalar meson. From the standpoint of charge symmetrical theory π meson must also be pseudoscalar. Then it is concluded that both proton and neutron are spinor (pseudospinor) fields. The determination of the spinor character of $\mu - \mu_0$ meson may be obtained by the μ^- capture and $\mu \to e + 2\mu_0$ decay discussed in (A). But if μ_0 meson mass is small, this cannot be detected by experiments.

Admitting as above that both proton and neutron are the same spinor fields, our selection rule about the τ-meson decay process[8] remains unaltered and pseudoscalar π meson is the most reasonable in all respects.

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On Mixed Field Theories and Vacuum Polarization

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The gauge invariance of the induced current in the vacuum by the presence of an electromagnetic field, has been investigated by many

authors from the standpoint of the mixture theory which was first proposed by Sakata and Pais. However, their calculations have been confined to the order e^2 in the case of charged scalar meson, spinor and vector meson fields. According to their results the photon self-energy vanishes provided

$$n_0-2n_{1/2}+3n_1=0 \tag{I}$$

$$\sum_{j=1}^{n_0}(m_0^{(j)})^2-2\sum_{j=1}^{n_{1/2}}(m_{1/2}^{(j)})^2+3\sum_{j=1}^{n_1}(m_1^{(j)})^2=0, \tag{II}$$

where n_0, $n_{1/2}$ and n_1 denote the number of charged scalar meson, spinor and vector meson fields which interact simultaneously with the external electromagnetic field and $m_0^{(j)}$, $m_{1/2}^{(j)}$ and $m_1^{(j)}$ the corresponding rest masses. But cancellation of the divergences which arises in the gauge invariant terms in the current density is not achieved by any similar method of renormalization.[1]

In this letter we investigate the gauge invariance of the vacuum polarization in the second order approximation by introducing the higher spin fields besides those considered above and the more complicated interaction with the electromagnetic field than those in the current one. Here we use effectively the method of the quantum theory of the generalized local field.[2]

In order to introduce the higher spin fields, first we assume that the wave equation for a particle of arbitrary spin in the absence of interaction is the following linear differential equation of the first order with matrix coefficients γ_μ:

$$(\gamma_\mu\Box_\mu+m)\psi=0, \quad \left(\Box_\mu=\frac{\partial}{\partial x_\mu}, \ \mu=1,2,3,4\right). \tag{1}$$

Besides, two additional physical assumptions concerning the particle are made, namely, (i) that it satisfies the usual secnod order wave equation with a fixed value of the rest mass m:

$$(\Box^2-m^2)\psi=0 \ \left(\Box^2=\frac{\partial^2}{\partial x_\mu \partial x_\mu}\right) \tag{2}$$

and (ii) that either the total charge or the total energy for particle field is positive definite. (ii) is necessary if the field is to be quantized in the interaction free case as usual.

For simplicity we consider especially the particle with the maximum spin 3/2. According to H. Chandra[3] the matrices γ_μ which describe the spin properties of the particle with 3/2 spin are given by

$$\gamma_\mu=\alpha_\mu+i\omega\beta_\mu, \quad (\omega=\alpha_1\alpha_2\alpha_3\alpha_4) \tag{3}$$

where α_μ and β_ν commute with each other and satisfy the Dirac and the Duffin-Kemmer commutation relations respectively. Now there are two inequivalent irreducible representations of γ_μ corresponding to the 5-row and the 10-row representation of β_μ, for the former the matrices γ_μ are 20×20 (scalar representation) and for the latter 40×40 (vector representation). Then the charge and current of the vacuum polarization for the scalar representation takes the following form in the second order approximation e^2

$$\delta j_\mu^{(3/2)}(x)=-e^2\int_{-\infty}^{+\infty}K_{\mu\nu}^{(3/2)}(x'-x)A_\nu(x')d^4x', \tag{4}$$

where

$$\begin{aligned}K_{\mu\nu}^{(3/2)}(x)=\frac{4}{m^4}\{&\Box_{\mu\rho\sigma}\Delta^{(1)}\cdot\Box_{\nu\rho\sigma}\bar{\Delta}+\Box_{\nu\rho\sigma}\Delta^{(1)}\cdot\\ &\Box_{\mu\rho\sigma}\bar{\Delta}-\delta_{\mu\nu}\Box_{\rho\sigma\tau}\Delta^{(1)}\cdot\Box_{\rho\sigma\tau}\bar{\Delta}-\\ &-\Box_{\mu\nu\rho}\Delta^{(1)}\cdot\Box_{\rho\sigma\sigma}\bar{\Delta}+\\ &+m^2(\Box_{\mu\nu\rho}\Delta^{(1)}\cdot\Box_\rho\bar{\Delta}+\Box_{\mu\nu}\Delta^{(1)}\cdot\Box^2\bar{\Delta}-\\ &-\delta_{\mu\nu}\Box_{\rho\sigma}\Delta^{(1)}\cdot\Box_{\rho\sigma}\bar{\Delta})-m^4\Box_{\mu\nu}\Delta^{(1)}\cdot\bar{\Delta}\}+\\ &+8\{\Box_\mu\Delta^{(1)}\cdot\Box_\nu\bar{\Delta}+\Box_\nu\Delta^{(1)}\cdot\Box_\mu\bar{\Delta}-\\ &-\delta_{\mu\nu}(\Box_\rho\Delta^{(1)}\cdot\Box_\rho\bar{\Delta}+m^2\Delta^{(1)}\cdot\bar{\Delta})\}\end{aligned} \tag{5}$$

with the abbreviation of $\Box_{\mu\nu\rho}=\frac{\partial^3}{\partial x_\mu \partial x_\nu \partial x_\rho}$.
The condition for the gauge invariance of (4) is

$$\Box_\nu K_{\mu\nu}^{(3/2)}(x)=0, \tag{6}$$

while in fact

$$\Box_\nu K^{(3/2)}_{\mu\nu}(x) = -12\delta^{(4)}(x)\cdot\Box_\mu\Delta^{(1)}(x). \quad (7)$$

From (7) it is evident that the photon self-energy vanishes also when a general mixture of each field with spin 0, 1/2, 1 and 3/2 respectively is considered provided

$$n_0 - 2n_{1/2} + 3n_1 - 6n_{3/2} = 0 \qquad (\mathrm{I}')$$

$$\sum_{j=1}^{n_0}(m_0^{(j)})^2 - 2\sum_{j=1}^{n_{1/2}}(m_{1/2}^{(j)})^2 + 3\sum_{j=1}^{n_1}(m_1^{(j)})^2 - 6\sum_{j=1}^{n_{3/2}}(m_{3/2}^{(j)})^2 = 0, \qquad (\mathrm{II}')$$

these equations are a generalization of the conditions (I) and (III). A detailed evaluation of (4) leads ultimately to the following expression :

$$\delta j_\mu^{(3/2)}(x) = \frac{3i}{2\pi}e^2\int_{-1}^{1}du\left[\frac{1}{z}\exp\{-\frac{i}{4}z(1-u^2)\Box^2\}\cdot \exp(im^2z)\right]_{(z=0)} A_\mu(x) - \frac{e^2}{\pi}\left(I_3+\frac{1}{6}I_1\right)J_\mu(x) + \frac{e^2}{\pi}\left(\frac{1}{4}I_2 - \frac{2}{15}\right)\left(\frac{\Box^2}{m^2}\right) J_\mu(x) - \frac{e^2}{\pi}\left(\frac{17}{420}I_1 + \frac{1}{70}\right)\left(\frac{\Box^2}{m^2}\right)^2 J_\mu(x) - \frac{e^2}{\pi}\frac{15003}{816480}\left(\frac{\Box^2}{m^2}\right)^3 J_\mu(x) - \cdots\cdots\cdots, \qquad (8)$$

where

$$I_n = \frac{1}{(im^2)^{n-1}}\int_{-\infty}^{+\infty}dz\cdot\frac{1}{z^n}\left(\frac{z}{|z|}\right)\exp(im^2z). \qquad (9)$$

An analogous expression to (8) is obtained for the vector representation. The introduction of the spin 3/2 field intensifies the difficulty in that it leads to a logarithmically divergent $(\Box^2)^2 J_\mu$ besides the quadratic divergent $\Box^2 J_\mu$. Unfortunately these infinities cannot be cancelled by the corresponding ones which occur in the case we have a charged vector meson field with vector and tensor couplings[4] for the signs in the two situations are the same. A fuller account will appear in a later issue of this journal.

1) H. Umezawa and R. Kawabe: Prog. Theor. Phys. **4** (1949), 369, D. Feldman: Phys. Rev. **76** (1949), 298.

2) S. Kanesawa: Prog. Theor. Phys. **5** (1950), 157.

3) H. Chandra: Proc. Roy. Soc. **192** (1948), 195.

4) D. Feldman: confer (1).

Errata: Meson Production by X-Ray

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April 26, 1950

In our previous letter[1] two serious mistakes have been committed, which we hasten to correct in this note. (A fuller account will be published in time.)

In the first place, the value of $r_0^2 = (\hbar/\varkappa c)^2\cdot 10^{-2}$ given there was erroneous[2] and must be multiplied by a factor $(1/2\pi)^2$, so that its correct value turns out 1.8×10^{-28}. Consequently one has to put $f^2/4\pi\hbar c = 1\sim 10$ in order to obtain the experimental value of the total cross section (per nucleon) $10^{-29}cm^2$ according to the calculation in the order e^2f^2. (Higher order approximations are now being investigated.)

In the second place, our judgement on the angular distribution of the produced pi-mesons was too premature. The detailed report[3] on the experiment explains that the factor $\sin^2\theta$ is due to the arrangement of the meson source and the detector (plates) and that the mesons can be considered coming out *isotropically*. In this respect one would be inclined to prefer a pseudoscalar pi-meson to a scalar one, because the latter gives a $\sin^2\theta$ distribution while the former is produced almost independent of the angle near the threshold. It will be appropriate, however, to refrain from drawing any decisive conclusion until one will have inquired into more detailed circumstances.

We may also add that the curve given in the previous letter cannot be compared directly with the experimental results, because it

epresents the cross-section for a monochromatic incident beam, whereas the actual experiment was performed by a continuous X-ray, although the qualitative feature of the curve will not be very much different for both cases. Also other circumstances are to be taken into account before comparing theoretical predictions with experimental results.[4)]

1) Z. Koba, T. Kotani and S. Nakai: Prog. Theor. Phys. **5** (1950), 137.
2) We are very much indebted to Messieurs Fujimoto, Yamaguchi and other members of Tokyo University of having pointed out the mistake and also for their critical discussions.
3(E M. McMillan, J. M. Peterson and R. P. White: Science **110** (1949), 579.
4) Y. Fujimoto, K. Nishijima, T. Okabayashi, K. Takayanagi and Y. Yamaguchi: Lecture at the semi-annual meeting of the Physical Society of Japan, April, 1950.

Cosmic-Ray Underground

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May 1, 1950

Good evidence was given previously by the computations of the soft component and the showers produced by μ-mesons at great depths[1)] for the appropriateness of taking account of merely the electromagnetic interaction of μ-mesons with matter on which basis the well-known bend appeared in the intensity *versus* depth curve of the cosmic-ray underground had been explained by Hayakawa and Tomonaga.[2)]

We intended here to give further supports to the abovementioned theory by calculating numerically the directional intensity and the temperature effect of the cosmic-ray underground.

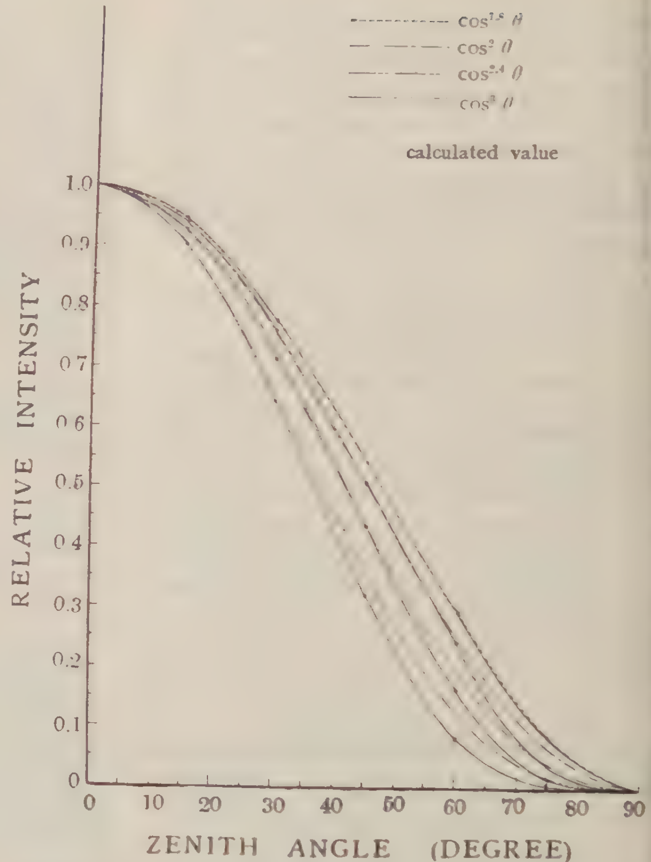
Of π-mesons produced high in the atmosphere by the primary rays, most of soft ones decay into μ-mesons while in flight and hard ones survived until the sea-level are soon absorbed after penetrating into the ground. Then we assume that the electromagnetic interaction of μ-mesons play the important role in the ground.

The diffusion equation for obliquely incident π-mesons including their absorption in the atmosphere was solved using the energy spectrum of the primary rays determined by Hayakawa and Nishimura[3)] as a part of the source function. Then the differential energy spectrum of μ-mesons was obtained on the basis of π-μ-decay. Next, the integral energy spectrum of μ-mesons on the sea-level was computed numerically for every 15° of the zenith angle of incidence θ assuming the masses of the π- and μ-meson and the life-time of the π-meson to be 286m, 217m (m; electron mass), and 1×10^{-8} sec, respectively.

The directional intensity: We have the intensity for given direction and depth by inserting for the energy appeared in the integral energy spectrum on the sea-level corresponding to that direction the value derived from the energy-range relation of μ-mesons in the ground to be penetrable to the depth under consideration.

The results of the calculations are shown in the figure in which the solid curves are drawn through calculated points marked by dots corresponding downwards to the depth of 200, 730, 1700, 3000 m w. e. (water equivalent) respectively and the four kinds of dotted curves give $\cos^n\theta$ for $n=1.8, 2.0, 2.4, 3.0$ downwards respe tively. For 200 m w. e. the angular distribution is exactly $\cos^{1.8}\theta$ reflecting the energy spectrum of π-mesons; for 730 m w. e. it may be approximated as $\cos^2\theta$ and is a little sharper than the experimental result $\cos^{1.7}\theta$ of Barnóthy and Forró[4)]; and for 1700 m w. e. it is about $\cos^{2.4}\theta$ and this is in fairly good accord with the Greisen's experimental result of about $\cos^3\theta$[5)] when we take the experimental error into account.

Although we have so far only one available experiment of Greisen at 1700 m w. e. to testify crucially whether the deviation from the $\cos^{1.8}\theta$ law takes place in the deeper region, his result may serve, if it is really true, as a strongest evidence for the validity of the theory of Hayakawa and Tomonaga.[1),2)] Thus we may conclude that the cosmic-ray far underground consists of μ mesons and that they interact with matter according to the ordinary quantum electrodynamics. Further the Greisen's opinion[6)] that the cosmic-ray phenomena underground can well be explained by the neutrino hypothesis of Barnóthy and Forró seems highly improbable.



Solid curves represent downwards the calculated results for $x=200$, 730, 1700, 3000 m w. e., respectively.

The temperature effect: The temperature coefficient of the hard component in the ground referring to the outer-air temperature was found to be +0.74% per °C for the depth of 1000 m w. e. according to the experiment of Barnóthy and Forró.[4),7)] But simple calculations of Forró[7)] and one of the authors[8)] had shown that this coefficient can not be larger than +0.16% per °C. We examined here how much approach toward the experimental result could be made by the theory hitherto used. It turned out that the coefficient can not exceed +0.4% per °C. and exact agreement with experiment could not be obtained.

Moreover, the reason to explain the existing small numerical discrepancy between theory and experiment could not be thought of at present.

Detailed accounts will be seen at a later date.

1) S. Hayakawa and S. Tomonaga: Prog. Theor. Phys. **4** (1949), 496.

2) S. Hayakawa and S. Tomonaga: Prog. Theor. Phys. **4** (1949), 247.

3) S. Hayakawa and J. Nishimura: Prog. Theor. Phys. **4** (1949), 232; Journ. Sci. Res. Inst. **44** (1949), 47.

4) J. Barnóthy and M. Forró: Phys. Rev. **74** (1948), 1300.

5) K. Greisen: Private communication to Y. Miyazaki.

6) K. Greisen: Phys. Rev. **76** (1949), 1718.

7) M. Forró: Phys. Rev. **72** (1947), 868.

8) T. Miyazima: Prog. Theor. Phys. **3** (1948) 99.

Structure of Electron in λ-Process

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May 1, 1950

The formalism of Dirac's λ-limiting process[1),2)] gives us a converging result in classical point electron, but fails in quantum theory. We want to clarify the main device of the source of convergence, and seek for the possible modification allowed to remove other types of divergences. With this aim in view, we firstly reinterpret it as a model of electron. The Schwinger-Tomonaga equation is (the integrability is only proved for flat surface and $\vec{\lambda}=0$)

$$H(x)\Psi[\sigma]=S_\mu(x)A_\mu(x)\Psi[\sigma]$$
$$=i\frac{\partial}{\partial\sigma(x)}\Psi[\sigma] \qquad (1)$$

and the second order interaction Hamilton density is (classical)

$$H^{(2)}=\frac{1}{2}\int_\sigma A_\mu^W(x')\,i\Big(N'_\rho\frac{\partial}{\partial x'_\rho}\Big)[A_\mu(x'),\ A_\nu(x)]_- S_\nu(x)dF' \qquad (2)$$

where $\Box A_\mu^W=S_\mu$, and λ-process gives

$$i\Big(N'_\rho\frac{\partial}{\partial x'_\rho}\Big)[A_\mu(x'),\ A_\nu(x)]_-$$
$$=-\delta_{\mu\nu}\Big(N'_\rho\frac{\partial}{\partial x'_\rho}\Big)\frac{1}{2}(D(x'-x+\lambda)+D(x'-x-\lambda))$$
$$=\delta_{\mu\nu}\Big(1+(N'_\rho\lambda_\rho)\frac{\partial}{\partial(N'_\rho\lambda_\rho)}\Big)\frac{D(x'-x+\lambda)-D(x'-x-\lambda)}{2(N'_\rho\lambda_\rho)}. \qquad (3)$$

Taking flat surface and $\vec{\lambda}=0$;

$$=\delta_{\mu\nu}\Big(1+\lambda_0\frac{d}{d\lambda_0}\Big)\Big(1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}\Big)\frac{1}{(2\pi)^3}\int e^{iK(x'-x)}\Big(\frac{\sin k\lambda_0/2}{k\lambda_0/2}\Big)^2 dK \qquad (4)$$
$$=\delta_{\mu\nu}\Big(1+\lambda_0\frac{d}{d\lambda_0}\Big)\Big(1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}\Big)\int\frac{\delta(|x'-x''|-(\lambda_0/2))}{4\pi(\lambda_0/2)^2}\frac{\delta(|x''-x|-(\lambda_0/2))}{4\pi(\lambda_0/2)^2}dx''\ (\lambda_0>0) \qquad (5)$$

Thus, defining the field produced by the currents $S_\mu^{(1)}$, $S_\mu^{(2)}$, $S_\mu^{(3)}$;

$$S_\mu^{(1)}(x)=\int S_\mu(x-x')\frac{\delta(|x'|-(\lambda_0/2))}{4\pi(\lambda_0/2)^2}dx';$$
$$\Box A_\mu^{(1)}=S_\mu^{(1)}$$
$$S_\mu^{(2)}(x)=\int S_\mu(x-x')\lambda_0\frac{d}{d\lambda_0}\frac{\delta(|x'|-(\lambda_0/2))}{4\pi(\lambda_0/2)^2}dx';$$
$$\Box A_\mu^{(2)}=S_\mu^{(2)}$$
$$S_\mu^{(3)}(x)=\int S_\mu(x-x')\Big(2\lambda_0\frac{d}{d\lambda_0}+\frac{1}{2}\lambda_0^2\frac{d^2}{d\lambda_0^2}\Big)\frac{\delta(|x'|-(\lambda_0/2))}{4\pi(\lambda_0/2)^2};\quad \Box A_\mu^{(3)}=S_\mu^{(3)}. \qquad (6)$$

We have for interaction Hamiltonian;

$$H^{(2)}=\frac{1}{2}\int\{S_\mu^{(1)}A_\mu^{(1)}+S_\mu^{(2)}A_\mu^{(2)}+$$
$$+(S_\mu^{(3)}A_\mu^{(1)}+S_\mu^{(1)}A_\mu^{(3)})\}dx. \qquad (7)$$

This represents that the electron composed of three sorts of charge in λ-process: the two are the independent currents $S_\mu^{(1)}$ (which is exactly that of Lorentz-Shell model with radius $\lambda_0/2$) and $S_\mu^{(2)}$ which do not act mutually, the other is current $S_\mu^{(3)}$ which does only mutual interaction with current $S_\mu^{(1)}$. Schematically shown in Fig. 1. Thus, the self-energy of electron comes from the self-energy of the currents $S_\mu^{(1)}$ and $S_\mu^{(2)}$ plus mutual interaction energy of currents $S_\mu^{(3)}$ and $S_\mu^{(1)}$. The latter is negative and the total self-energy vanishes.

Formally this is the consequence of the operator $1+\lambda_0\frac{d}{d\lambda_0}$ in (5), since by rewriting it in the following form;

$$\bar{H}^{(2)}=\Big(1+\lambda_0\frac{d}{d\lambda_0}\Big)\Big(1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}\Big)$$
$$\times\frac{1}{2}\int S_\mu^{(1)}A_\mu^{(1)}dx \qquad (8)$$

this model may be reinterpreted as the Lorentz-Shell model, and so

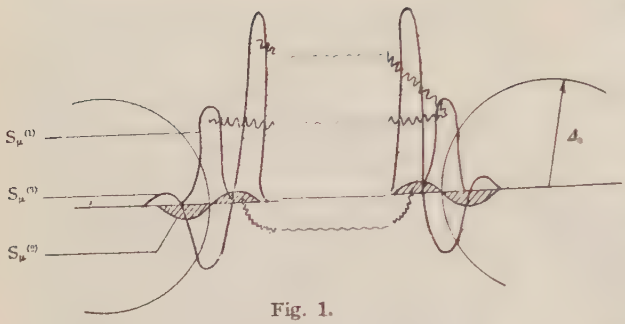

Fig. 1.

$$\bar{H}_{\text{coulomb}}=\left(1+\lambda_0\frac{d}{d\lambda_0}\right)\left(1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}\right)\frac{e^2}{\lambda_0}=0 \tag{9}$$

It should be remarked that the quantum theoretical divergence which appears in the one-electron theoretical self-energy can be removed by the change of the model; the subsequent operator $1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}$ removes the term of the form $1/\lambda_0^2$, but in the Lorentz-Shell model gives logarithmically diverging result for quantum self-energy $\int^\infty k\,dk$, the application of this operator does not remedy this divergence, but it is done if one takes more decisive cut off, for example $e^{-k\lambda_0}$,

$$H^{(2)}_{\text{fluct}}\propto\left(1+\lambda_0\frac{d}{d\lambda_0}\right)\left(1+\frac{1}{2}\lambda_0\frac{d}{d\lambda_0}\right)\times\int_0^\infty ke^{-k\lambda_0}dk=0. \tag{10}$$

In hole theory we must amplify $S_\mu^{(3)}$ in order to get the finite results, because in this case the electron spreads itself due to Fermi statistics,[3] and so the mutual interaction energy of $S_\mu^{(3)}$ with $S_\mu^{(1)}$ becomes insufficient to cancel the self-energy of $S_\mu^{(1)}$ and $S_\mu^{(2)}$. This amplification can be made without affecting any real features of the theory by the following;

$$i\left(N'_\rho\frac{\partial}{\partial x'_\rho}\right)\cdot[A_\mu(x')A_\nu(x)]=\delta_{\mu\nu}\left(1+f(\lambda_\mu^2)(N'_\rho\lambda_\rho)\frac{\partial}{\partial(N'_\rho\lambda_\rho)}\right)\frac{D(x'-x+\lambda)-D(x'-x-\lambda)}{2(N'_\rho\lambda_\rho)} \tag{11}$$

where

$$f(\lambda_\mu^2)\sim a\log\lambda_0+b+c\lambda_0\cdots \quad (\text{for small } \lambda_0). \tag{12}$$

The restriction (12) ensures that only high energy photon amplitude of order $1/\lambda_0$ is modified. Inside this restriction, the self-energy of hole-theory can be made converge by taking

$$f(\lambda_\mu^2)=\frac{1}{2}\log\frac{1}{(-\lambda_\mu^2)M^2}+(\text{power of }\sqrt{-\lambda_\mu^2})\ . \tag{13}$$

1) P. A. M. Dirac, Proc. Roy. Soc. **A 180** (1942), 1.
2) W. Pauli, Rev. Mod. Phys. **15** (1943), 75.
3) V. F. Weisskopf, Phys. Rev. **56** (1939), 72.

The Fast Protons in π^--Meson Stars

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May 5, 1950

Previously we discussed the new modes of meson-capture[1]:

$$\pi^-+N+P\to N+N, \tag{nn}$$
$$\pi^-+P+P\to N+P. \tag{np}$$

If we adopt the Fermi gas model for a nucleus, the brode momentum distribution of nucleons inside the nucleus leads to the brode energy distribution (up to $\mu c^2+E-V=138$ Mev., E=Fermi energy and V=well depth =30 Mev.) of the final nucleons of capture processes (nn) and (np), which is consistent with the existence of fast protons in meson stars.[2] This energy distribution was not taken into account in our previous calculations.[1]

The probability that the two "primary nucleons" of π^--meson capture process have energy E_1 and E_2, respectively, is given by the following expression:

$$f(E_1,E_2)\propto\iiiint d\boldsymbol{P}_1^0 d\boldsymbol{P}_2^0 d\boldsymbol{P}_1 d\boldsymbol{P}_2$$
$$|\boldsymbol{P}_1^0|,\ |\boldsymbol{P}_2^0|\leq P_F$$
$$|\boldsymbol{P}_1|,\ |\boldsymbol{P}_2|\geq P_F$$

$$\cdot\delta(P_1+P_2-P_1^0-P_2^0)$$
$$\cdot\delta\left(\frac{P_1^2}{2M}+\frac{P_2^2}{2M}-\mu c^2-\frac{P_1^{02}}{2M}-\frac{P_2^{02}}{2M}\right)$$
$$\cdot\delta\left(\frac{P_1^2}{2M}-E_1\right)\delta\left(\frac{P_2^2}{2M}-E_2\right)$$
$$\cdot(N.F.)^2,$$

where P_F = Fermi momentum $=\sqrt{2ME_F}$,
M = nucleon mass,
and μ = meson mass.

Of course

$$f(E_1, E_2)=f(E_2, E_1).$$

(N.F.) represents the matrix element of nuclear force and it can be well approximated to be constant. Then we get the following result :

For $E_1>E_2$, $f(E_1,E_2)=C\times$

$$[2\sqrt{E_1E_2}-(\mu c^2-E)]^{3/2}\cdots \qquad \frac{1}{2}(\sqrt{\mu c^2}+\sqrt{E})^2>E_1>E_+, \qquad (1)$$

$$[\{2E-(\sqrt{E_F}-\sqrt{E-E_F})^2\}^{3/2} +3(2E_F-E)\{\sqrt{E_F}+\sqrt{E-E_F}-\sqrt{E_1}+\sqrt{E_2}\}]\cdots \qquad E_+>E_1>E_-, \qquad (2)$$

$$[\{2E-(\sqrt{E_F}+\sqrt{E_F-E})^2\}^{3/2}-\{2E-(\sqrt{E_F}-\sqrt{E_F-E})^2\}^{3/2}+\{2\sqrt{E_1E_2}-(\mu c^2-E)\}^{3/2}+6(2E_F-E)\sqrt{E_F}]\cdots \qquad E_->E_1>\frac{E+\mu c^2}{2}, \qquad (3)$$

where C = normalization constant to be chosen

$$\int f(E_1, E_2)\,dE_1dE_2=1,$$

and

$$E=E_1+E_2-\mu c^2,$$
$$E_\pm=\frac{1}{2}\left[E+\mu c^2+\sqrt{(E+\mu c^2)^2-\{\mu c^2\mp2\sqrt{E_F(E-E_F)}\}^2}\right].$$

If $E_F>E>0$, the cases (2) and (3) do not exist. The relative values of $f(E_1, E_2)$ are shown in Fig. 1. The energy spectrum of

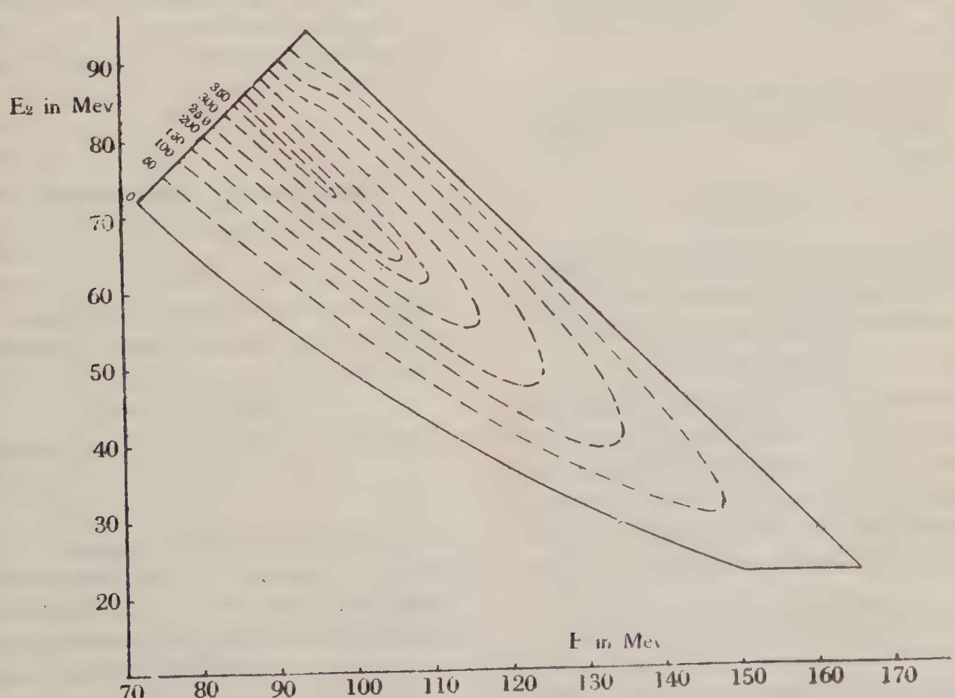

Fig. 1. relative value of $f(E_1, E_2)$

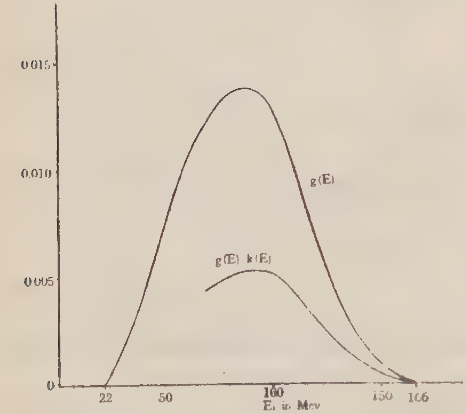

Fig. 2.

a fast nucleon regardless the energy of its partner is given by

$$g(E)=\int f(E, E')\,dE',$$

which is shown in Fig. 2. E means energy of a proton measured inside nuclear matter and the actual energy outside the nucleus is $E-V=E-30$ Mev. (V=well depth). Then the energy spectrum $h(E)$ of primary fast proton is given by:

I) Scalar meson with scalar coupling or Pseudoscalar meson with pseudoscalar coupling

$$h(E)=\frac{1}{2}g(E).$$

II) Assuming the pseudoscalar meson theory with pseudoscalar coupling and using the phenomenological nuclear potential :[3]

$$h(E)=0.47g(E).$$

In order to get the actual spectrum of fast protons, we must multiply $h(E)$ by the probability $k(E)$ for escaping out of the nucleus without nuclear collisions. If we assume the absorption of π^--meson occurs with equal probability inside the nucleus, then

$$k(E)=\frac{3}{2}\int_0^{2R}\left\{1-\left(\frac{\rho}{2R}\right)^2\right\}e^{-\frac{\rho}{\lambda(E)}}\frac{d\rho}{2R}$$

where $\lambda(E)$ is the collision mean free path of a fast nucleon with energy E inside nuclear matter[4] and R means the nuclear radius.

Fig. 2. also shows the curve $g(E)k(E)$ (and not $h(E)k(E)$) for a heavy nucleus with mass number $A=100$.

In order to discuss this problem in more details, we must take account of the meson stars initiated from light nuclei in photographic emulsion.

Thus we can explain the existence of fast protons in meson stars without the assumption of recoil trition.[5]

The authors express their hearty thanks to Mr. Takeda for his helpful discussions.

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2) F. L. Adelman, private communication. W. B. Cheston and L. Goldfarb; Bull. Am. Phys. Soc. **25**, No. 1 (1950), 12.
3) In this case we used the same device as adopted by Foldy and Marshak in their calculation of meson production, Phys. Rev., **75** (1949), 1493.
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5) S. Tamor; Phys. Rev., **77** (1950), 412 L.

A Model for the Condensation Phenomena

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May 15, 1950

Let us consider the system composed of N molecules contained in the vessel whose volume is V. We divide V in equal cells, and denote the number of cells which contain i particles by m_i. The potential energy of each cell is assumed to depend only on the number of molecules in it, and is rep-

resented by $i\epsilon_i$. Neglecting the energy of interaction between different cells, the configurational part of the partition function is written as follows:

$$Q_\tau = m!\, N!\left(\frac{V}{m}\right)^N \sum_\alpha \prod_{i=0}^{N} \frac{\exp(-i\epsilon_i m_i/kT)}{m_i!\,(i!)^{m_i}}, \tag{1}$$

where $\sum_\alpha$ means the summation with regard to all m_i's under the restriction of $\sum_{i=0}^{N} m_i = m$, $\sum_{i=0}^{N} i m_i = N$. The set of m_i in the greatest term in $\sum_\alpha$ is,

$$m_i = Ax_i\xi^i, \quad x_i = \exp(-i\epsilon_i/kT)/(i!).$$

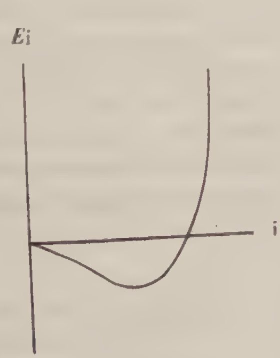
A and ξ are the indeterminate constants of Lagrange, and ξ is determined from

$$N/m = (\sum_i i x_i \xi^i)/(\sum_i x_i \xi^i). \tag{1}$$

ξ represents fugacity.

We may expect qualitatively that $\epsilon_i \to -0$ in the infinite dilution ($i \to 0$), and $\epsilon_i \to +\infty$ when the density in the cell is very large, and ϵ_i has negative value in some intermediate range of i.

Then we can show: when ξ is sufficiently small or sufficiently large, m_i has one sharp maximum against i, and when ξ is in some intermediate range, m_i has two sharp maxima against i. We may interpret it as follows: by an isothermal compression, a part of imperfect gas becomes liquid phase, and at last the whole becomes liquid.



If T is larger than some value, however, only one maximum appears; this value T is the critical temperature

The pressure p is

$$pV/kT = m \log \sum x_i \xi^i. \tag{3}$$

Remarking that $x_i\xi^i$ has one or two sharp maxima, we replace $\sum$ in (2) and (3) by integration, and we can conclude that ξ is monotonic increasing against i, and has the horizontal part in some range of N. Therefore, the isotherm is shown to have horizontal part without any thermodynamical operation.

The detailed discussions will be reported shortly in this journal.

Order of Magnitude Theory of the Depression of Coulomb Barrier Height in Highly Excited Nuclei

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Formerly, Bagge[1] showed that the Coulomb barrier height $V(X)$ of a excited nucleus (atomic number$=Z$, mass number$=A$ and nuclear radius$=R=r_0A^{1/3}$, $r_0=1.37\times10^{-13}$ cm) decreases with increasing excitation energy X. He treated this phenomenon in greater details and gave $V(X)$ in the form of a infinite series. In so far as a tendency of $V(X)$ is concerned, we can readily derive it in a very simple but crude manner as follows:

Let a denote the average amplitude of surface wave of a nucleus with excitation energy X. Then, X would be proportional to the excess of surface energy:

$$4\pi a^2 O = \xi X \qquad (\xi = \text{const.}), \tag{1}$$

where

$$4\pi R^2 O = 4\pi r_0^2 O \cdot A^{2/3} = 14 A^{2/3} \ \mathit{Mev}.$$

means the surface energy part in the so-called Weiszäcker-Bethe's semi empirical formula (see Bohr-Wheeler. ref. 2). As a result of

the surface wave, the Coulomb barrier height decreases to the smaller value

$$V \sim Ze^2/(R+a) = \frac{V_0}{1+\frac{a}{R}} \qquad (2)$$

where $V_0 = \frac{Ze^2}{R}$ is the barrier height for the ground state. Inserting (1) into (2), we get the energy-dependence of barrier height.

If we introduce the temperature T of excited nucleus:

$$X \sim \frac{A}{10} T^2, \quad X,\ T \ in\ Mev$$

(this relation is experimentally confirmed by Perkins et al., ref. 3)), one finds

$$V(T) \sim \frac{V_0}{1 + A^{\frac{1}{6}} \left(\frac{\xi}{144}\right)^{\frac{1}{2}} T}.$$

For instance, we consider the nucleus $_{35}Br^{80}$, which was treated by Bagge. If we tentatively take ξ as 1/3, we find the relation

$$V(T) \sim \frac{V_0}{1+0.1\,T}.$$

This very crude formula gives the results in good agreement with the ones obtained by Bagge.

The depression of barrier height results in the modification of the energy spectrum of charged particles evaporated from highly excited nuclei in lower energy parts.

Thus we must not overlook this effect, when we want to discuss the large cosmic-ray stars.

1) E. Bagge; Ann. d. Phys. **33** (1938), 389.
2) N. Bohr and J. A. Wheeler; Phys. Rev. **56** (1939), 426.
3) J. B. Harding, S. Lattimore and D. H. Perkins: Proc. Roy. Soc. **196** (1949), 325.

Some Remarks on the Relativistic Quantum Field Theory

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June 3, 1950

In the relativistic quantum field theory developed by S. Tomonaga and J. Schwinger the properties of the fields are treated in the interaction representation, and the four-dimensional commutation relations are derived for the field quantities for the free fields. On the more, the Hamiltonian density in the Schrödinger equation is not reasonably derived from the Lagrangian density, and integrability condition for the Schrödinger equation is used to determine it.

We have studied on the relativistically invariant formulation of the theory in which all the time development of the system should be described in the Shrödinger representation and the Hamiltonian density should be reasonably determined by the physical relations.

The fundamental point of these theory is how to quantise the field in covariant form under Lorentz-transformation without referring to the functional form of the field quantities for the free field. We researched on the fundamental relation between the Schrödinger equation and the canonical relation, and obtained the reasonable commutation relations.

The Schrödinger equation we derived have the form

$$\left(H[P,\ \sigma] + \frac{\hbar}{i}\frac{\delta}{\delta\sigma_P}\right)\Psi[\sigma] = 0$$

$$H[P,\ \sigma] = -n^\mu(P)\, n_\nu(P)\, T_\mu{}^\nu$$

where $T_\mu{}^\nu$ are the ordinary four-dimensional energy-momentum-tensor, and $n^\mu(P)$, $n_\nu(P)$ are the component of the unit vector normal to the space-like surface σ at the point P. The Hamiltonian density is determined from the conditions that they should be invariant under arbitrary Lorentz-transformation and

should have close connection with the energy-momentum-vector which corresponds to the particle character of the fields.

As the above mentioned Schrödinger equation represents the change of the state vector along the surface normal at the point P, the canonical relation should have the direct connection with the derivatives of the field quantities with respect to the time coordinate in the tangential coordinate system at P. Then, we can define the canonical conjugate variables of the field variables ϕ^{μ} by the relations.

$$\pi_{\mu}(P,\ x)=\partial L\,/\,\partial\frac{\partial\phi^{\mu}(x)}{\partial\tau_{P}}$$

where L is the Lagrangian density and τ_P is the coordinate along the surface normal σ at P.

The commutation relations among these quantities have the ordinary form in the tangential coordinate system, and the four-dimensional commutation relations in the arbitrary coordinate system should be obtained by the Lorentz-transformation of the relations in the tangential coordinate system, and can be given by

$$[\phi^{\mu}(x_P),\ \pi_{\nu}(Q,\ x_Q)]$$
$$=\delta(x_P-x_Q)\sum_{\lambda}a_{\lambda}^{\mu}(P)a_{\nu}^{*\lambda}(Q),$$

$$[\phi^{\mu}(x),\ \phi^{\nu}(x_Q)]$$
$$=[\pi_{\mu}(P,\ x_P),\ \pi_{\nu}(Q,\ x_Q)]=0,$$

where δ is the ordinary δ-function, $a_{\lambda}^{\mu}(P)$ and $a_{\nu}^{*\lambda}(Q)$ are the coefficient of the transformation occurred in $\phi_{\mu}(x)$ and $\pi_{\nu}(Q,\ x)$ under the Lorentz-transformation between arbitrary Lorentz-system and the tangential coordinate system at P and Q respectively.

In these generalised theory, not only the Tomonaga's theory is included as the special case, we can treat the various problem concerning the interactions between the fields standing on the reasonable view point. Even in the case of vacuum or only one particle is presents, there exist various kinds of fields and interact each other in the virtual state, and the completely isolated field can not exist. Then, the problems of the self energy, vacuum polarisation and anomalous magnetic moment etc., can be treated reasonably in our pure Schrödinger representation, and even the wave function of a particle should be obtained after solving the problem in the case when only one particle presents apparently.

Detailed account of this work and related problems will appear in a later issue of this journal.

On the Production of γ-Rays accompanied by the Absorption of π-Meson by a Proton

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June 1, 1950

Recently, it has been reported that the π^- meson absorbed in the hydrogen gas produces high energy γ-rays, the spectrum of which probably has a flat maximum at 65Mev and is sharply cut at 130Mev.

This phenomenon is interpreted to be due to an elementary process in which a π^--meson is absorbed by a hydrogen nucleus, emitting at lea t two photons. In fact, the Bremsstrahlung of the π-meson is hardly effective because of its low kinetic energy. The γ-ray associated with the ordinary π-μ decay contributes only to magnitude of $e^2/\hbar c$ $(=1/137)$, and the energy of which is at most 30Mev (the difference of the rest energy between π-meson and μ-meson.)

As the life time that the π^--meson with ordinary kinetic energy ($\sim$14Mev), passing through the high pressure hydrogen gas, slows down and is captured into K-orbit of the hydrogen atom, is considerably short

compared with the life time of the ordinary decay,[1] we may consider only the probability that a π^--meson in K-orbit is absorbed by a proton. A part of the results of these calculations has already been reported by Marshak and Wightman.[2] We have checked their results and we give some detailed analysis. Though calculation is similar to theirs, our results obtained are slightly different.

In this paper, we take π^-- and π^0-meson to be both scalar or pseudoscalar with scalar- or pseudoscalar-type coupling. We have estimated the probabilities of the absorption of the π^--meson by a proton, i) accompanied by one photon emission (denoted by $W_{1\gamma}$), ii) accompanied by two photons emission (denoted by $W_{2\gamma}$) and iii) accompanied by a neutral meson emission (denoted by W_n). We have taken an approximation that in i) and ii) the terms less than $(\mu/M)^2$ and $(\mu/M)\ (\mu-\mu^0/\mu)$ are neglected compared with unity, and in ii) terms of order of (μ/M) are neglected.

The results of the calculation are represented in Table I in which $W_{2\gamma}$ and W_n are compared with $W_{1\gamma}$, taking μ to be equal to 280 electron mass. Before going to the discussion of the numerical results, we should note briefly the characteristics of the phenomena in each case above.

Case of i) ; the emitted photon has a sharp maximum at $\mu(1-\mu/2M)$ in spectrum. Assuming μ to be 280 electron mass, $\mu(1-\mu/2M)$ becomes 130Mev.

Case of ii) ; the spectrum of the emitted photons has the form $\{1-(1-2k/\mu)^2\}dk/\mu$ $0\leq k\leq\mu)$ in the scalar case and $\{1-(1-2k/\mu)^4\}\frac{dk}{\mu}$ in the pseudoscalar case respectively. Case of iii) ; As the angular distribution of the emitted neutral meson is isotropic, the γ-ray originating from the π^0-meson decay has an energy spectrum which has a center near $\mu/2$ and is uniform in the energy interval $P=[(\mu+\mu_0)(\mu-\mu_0)/1+\mu/M]^{1/2}$ and

$W_{1\gamma}(\text{scalar})=0.56\times10^{15}\ f_s^2/\text{sec}$ $\qquad$ $W_{1\gamma}(p.\ s.)=3.3\times10^{15}\ f_{p.s.}^2/\text{sec}$

$W_{2\gamma}/W_{1\gamma}(\text{scalar})=7.1\times10^{-3}$, $\qquad$ $W_{2\gamma}/W_{1\gamma}(p.\ s.)=2.0\times10^{-4}$

$W_n/W_{1\gamma}$

μ_0 in electron mass		275	270	260	250	210
p. s. $\pi^-\to$ p. s. π^0	$\bar{f}^2\times$	12	38	63	80	123
	$\bar{f}\Delta f\times$	1.78	5.4	8.3	9.8	10.6
	$\Delta f^2\times$	0.062	0.2	0.27	0.31	0.23
p. s. $\pi^-\to$ s. π^0	$\bar{f}^2\times$	0.047	1.5	6.7	13.8	49
	$\bar{f}\Delta f\times$	0.088	2.8	12.7	26	95
	$\Delta f^2\times$	0.040	1.3	5.9	12.2	45
s. $\pi^-\to$ s. π^0	$\bar{f}^2\times$	0.0^573	0.0^223	0.029	0.097	0.81
	$\bar{f}\Delta f\times$	0.061	1.9	8.8	18.2	68
	$\Delta f^2\times$	126	396	662	841	1300
s. $\pi^-\to$ p. s. π^0	$\bar{f}^2\times$	0.0^416	0.0^346	0.0^218	0.0^233	0.0^26
	$\bar{f}\Delta f\times$	0.0^342	0.013	0.055	0.105	0.27
	$\Delta f^2\times$	0.0^229	0.09	0.42	0.87	3.2
$\bar{P}$ in Mev		8.3	26	43	55	85

$$\bar{f}=\frac{1}{2}(f_{N^0}+f_{P^0}),\qquad \Delta f=\frac{1}{2}(f_{N^0}-f_{P^0}).$$

is zero outside the interval. If π^0-meson is of vector type, the spectrum of the γ-ray is considerably different from that due to the scalar π^0, μ, μ_0 and M are the masses of π^--meson, π^0-meson and nucleon, respectively.

Next, some discussions are presented about the results in Table I. As we can see, $W_{2\gamma}$ is only one percent in magnitude compared with $W_{1\gamma}$. Therefore, it is reasonable to consider that the Panofsky's experiment indicates the competition of the processes of the one photon emission with one neutral meson emission. Assuming that this interpretation is correct, it is concluded from Table I, with the reasonable value of the coupling constant, that the case of scalar π^- seems to be excluded. Panofsky's result, however, seems to indicate that the breadth of the spectrum is about 50Mev, having a flat maximum at 65Mev, which suggests about 250 electron mass for the π^0-meson. Further, it should be noted that the above discussion is not definite because of the uncertainty of the experiment itself. The γ-ray spectrum, for instance, due to the scalar π^0-meson decay has no maximum, while that due to the vector π^0-meson decay has a maximum. Detailed paper including the analysis of the vector-type meson will be reported in near future.

In conclusion, the authors should like to express their deep gratitude to Prof. S. Sakata for his valuable discussion and to Prof. M. Taketani who has kindly told us his information about Panofsky's results.

The constants $f_P{}^0, f_N{}^0$ represent the coupling constants of π^0 to the proton and neutron respectively.

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2) R. E. Marshak and A. S. Wightman, Phys. Rev. **76** (1949), 114.

Ground State of Deuteron according to Pseudoscalar Meson Theory

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June 7, 1950

The strength of the coupling between mesons and nucleons was usually determined by comparing the theoretically calculated binding energy of deuteron with experiment. However, the hitherto determination involved an ambiguity because of the too strong singularity of the two nucleon potential. It was found that the potential did not involve such a singularity so far as the pseudoscalar meson theory was concerned.[1),2),3)] In this letter we shall give a brief account for the result of calculation on the ground state of deuteron according to the symmetrical pseudoscalar meson theory on this new standpoint. Since there has been no evaluation of the coupling constant of pseudoscalar meson with nucleons, our result may give some contribution to the knowledge of this constant though the result is still too rough.

The customary potential in the ground state of deuteron is given by

$$W=-\frac{1}{2}g^2(1+\eta\Lambda)\frac{e^{-r}}{r} \tag{1}$$

in $\hbar=c=\mu=1$ units (μ=meson mass) where

$$\Lambda=3\ (\sigma^{(1)}\boldsymbol{X})(\sigma^{(2)}\boldsymbol{X})/r^2-1 \tag{2}$$

$$\eta=1+3/r+3/r^2 \tag{3}$$

It was shown that[3)] this was valid for large r and had no validity for $r<1$. When r is very large η is nearly equal to unity. When r approaches to unity W given by (1) may become too large. In this case η is nearly equal to 7. If we assume that the potential in the range of small r has little effect on the various properties of deuteron in its ground state we may obtain the effective potential by assuming an effective value of η.

If this assumption is valid it may be concluded from the above consideration that the effective value of η must be between 1 and 7. Then we can test whether the pseudoscalar theory can adequately describe or not the ground state of deuteron by examining these two extreme cases.

The numerical calculation is carried out according to the variational method. First we calculate the binding energy. Comparing it with the observed value (2.17 Mev) we determine the coupling constant (g^2) and the wave function. Next we calculate the electric quadrupole moment (Q) and the magnetic dipole moment (μ_D) of deuteron by making use of these coupling constant and wave function. The result is compared with experiment in the accompanied table,

	g^2	Q	μ_D
$\eta=1$	0.54	3.18	0.8577
$\eta=7$	0.140	2.72	0.8298
Obs.		2.73	0.8565

where Q is measured in 10^{-27} cm^2 and μ_D in nuclear magneton units. The numerical data which are used in the evaluation of the theoretical result are as follows: binding energy of deuteron $= 2.17$ Mev, $\mu = 286m$, nucleon mass$=1840m$ (m=electron mass), magnetic moment of proton$=2.7996$, that of neutron$=-1.9103$.

In view of the fact that the variational method can only give energy an approximate eigenvalue but does not give the wave function a sufficient approximation, the result may be considered to show that the theory and experiment are in agreement in this degree of approximation so far as the ground state of deuteron is concerned. Of course, in order to obtain the more definite conclusion it is necessary to calculate using a more exact form of the potential and according to a more exact method of solution.

A more detailed account will shortly be given in a later issue of this journal.

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2) G. Araki, Phys. Rev., **75** (1949), 1101, 1262; Prog. Theor. Phys., **4** (1949), 193.
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IN

COMMEMORATION OF

THE FIFTEENTH ANNIVERSARY

OF

THE DISCOVERY OF MESON THEORY

Hideki Yukawa

Preface

As early as 1935, Prof. H. Yukawa published in that year's issue of *Proceeding of Physico-Mathematical Society of Japan* (**17** (1935), 48) his celebrated paper '*On the Interaction of Elementary Particles, I*' in which he first proposed a far-reaching idea to explain the characteristic properties of nuclear forces. The true meaning implied in this paper has become more and more clear in the subsequent researches in the field of the theory of elementary particles.

According to his idea, the nuclear force field with the limited range about 10^{-13} cm should be described by the field equation of the type

$$\Box U - \varkappa^2 U = \rho$$

with $\varkappa \simeq 10^{13}$ cm^{-1}. Because, the corresponding Green's function to this equation is $\exp(-\varkappa r)/r$, which is just suited to represent a force not appreciably beyond the order $\varkappa^{-1}$. On the other hand, the source density ρ was related to the transition between the states of nuclear particle, and the existence of the charged field was also expected in the case of transition from proton state to neutron one or *vice versa*, which seems to be necessary to explain the saturation property of nuclear forces.

Furthermore, he suggested on the ground of the quantum field theories, which had been developed up to that time, the existence of new particles, charged or neutral, with mass

$$\mu = \hbar\varkappa/c$$

which is about 200 times electron mass. In connection with the theory of β-decay, he further assumed this new particle can decay to an electron-neutrino pair, that is

$$U^{\pm} \rightarrow e^{\pm} + \nu.$$

It follows as a natural consequence of this assumption that the new particle has a finite life time. This life time was later estimated to be about 10^{-6} sec.

On the discovery of new particles with intermediate mass between those of electron and proton in cosmic radiation in 1937, Prof. H. Yukawa's idea suddenly began to attract the attention of many theoretical physicists, and the refinement of his theory has been attempted by himself and many other authors. The name *meson* or *mesotron* was proposed for such new particles and corresponding denomination of *meson field* for its wave aspect.

The various covariant properties of meson field have been proposed by Prof. Yukawa and his collaborators, Prof. S. Sakata and Dr. M. Taketani, and by several other authors independently. The pseudoscalar field corresponding to meson with spin 0, and the vector field of spin 1, both obeying Bose statistics, seemed to be most preferable. However, the experimental researches with more refinement have become to show many discrepancies between the properties expected theoretically and those of mesons found in cosmic radiation. Especially, for the life time of mesons and the scattering cross-section of mesons by nucleons these discrepancies had been seemed to be fatal.

It must, however, be emphasized that the efforts devoted to various trials to find the way out of these difficulties have made remarkable progress in the development of the quantum field theory or the theory of elementary particles. In fact, it seems to be no

exaggeration to say that the theory of elementary particles has first settled its autonomy at the time of the discovery of meson theory.

It was, nevertheless, the final triumph of the meson theory when the so called π-meson was discovered, though its existence had been theoretically expected by Prof. S. Sakata and Prof. Y. Tanikawa since 1942, by the skilful analysis of the absorption and annihilation of mesons in matter and the refined experiments with photographic emulsion in 1947, just twelve years after it was predicted by Prof. Yukawa. Here, we wish to offer our congratulation to Prof. Yukawa on his admirable success in his theoretical researches and his honour of receiving the Nobel Prize in 1949 in recognization of his merit of the discovery and development of the meson theory.

It is, at present, generally believed that the π-meson, whose properties have been investigated by the analysis of its tracks in photographic emulsion and the artificial production of them in the laboratories in America, should be identified with mesons predicted by Prof. Yukawa, whereas those which are found in the hard component of cosmic radiation and now called μ-mesons are the decay products of π-mesons. There yet remain, however, many problems unsolved concerning with the meson theory, for example, the nature of neutral mesons and that of other kinds of mesons which seem to exist in cosmic radiation. Moreover, the meson theory, needless to say, shares with quantum electrodynamics the well-known defects which are inherent to the present quantum field theory. Though the state of things has been much improved by the recent discovery of the elegant covariant formalism and the artificial subtraction method for the remedy of so-called divergence difficulties in the quantum field theory, it seems to be doubtless that there yet remain essential discrepancies in the meson theory which are to be solved in the future development of the theory of elementary particles. It may, nevertheless, be expected that the meson theory will play a predominant role as the critical touchstone in the future theory.

In closing this preface I wish to thank to the authors who contributed valuable papers to this commemorial number. It is also my pleasant duty to express, as the representative of the editors of our journal, our cordial thanks to the authors who kindly took trouble of sending their papers from abroad.

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July 20, 1950.

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